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FULL LENGTH PAPER

Multi-item lot-sizing with joint set-up costs

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Abstract We consider a multi-item lot-sizing problem with joint set-up costs and constant capacities. Apart from the usual per unit production and storage costs for each item, a set-up cost is incurred for each batch of production, where a batch consists of up to C units of any mix of the items. In addition, an upper bound on the number of batches may be imposed. Under widely applicable conditions on the storage costs, namely that the production and storage costs are nonspeculative, and for any two items the one that has a higher storage cost in one period has a higher storage cost in every period, we show that there is a tight linear program with $O(mT^2)$ constraints and variables that solves the joint set-up multi-item lot-sizing problem, where m is the number of items and T is the number of time periods. This establishes that under the above storage cost conditions this problem is polynomially solvable. For the problem with backlogging, a similar linear programming result is described for the uncapacitated case under very restrictive conditions on the storage and backlogging costs. Computational results are presented to test the effectiveness of using these tight linear programs in strengthening the basic mixed integer programming formulations of the joint set-up problem both when the storage cost conditions are satisfied, and also when they are violated.

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1 Introduction

In this paper we consider a multi-item lot-sizing discrete time problem in which production takes place in (a possibly limited number of) mixed batches of constant capacity. Apart from the usual unit storage and production costs, there is a fixed cost per batch representing the use of limited capacity resources. An alternative interpretation involves a warehouse/retailer and decisions on the use of trucks of a given capacity to ship from the warehouse so as to (possibly) stock items and satisfy demand forecasts at the retailer. The objective is to find replenishment decisions for all items that satisfy the demand over a finite planning horizon, and minimize the sum of fixed and variable production costs and storage costs.

As explained below, the complexity of the multi-item lot-sizing joint set-up cost problem is determined by the specific assumptions made on the cost parameters. In this paper we develop a *compact* linear program, i.e., a linear programming formulation for the problem whose size is polynomial in the size of the input. Moreover, we identify widely-applicable conditions on the cost parameters under which, the proposed linear programming formulation is *tight*, i.e., it generates an optimal solution to the multi-item lot-sizing joint set-up cost problem.

More specifically, we first assume, without loss of generality, that the per unit costs have been normalized so that the production costs are zero over time, see for instance p. 132 in Pochet and Wolsey [16]. Then, we make the following assumptions on the per unit storage cost h_t^i for item i in period t

(i) *Nonspeculative* (Wagner–Whitin) cost condition: $h_t^i \geq 0$ for all i, t, and
(ii) *Dominance* condition: it is possible to index the items such that they have non-increasing storage costs in each period, that is, $h_t^i \geq h_t^{i+1}$ for all i, t.

Under the above conditions on the storage cost parameters, we show that an optimal solution for the multi-item lot-sizing joint set-up cost problem can be obtained by solving a linear program with $O(mT^2)$ constraints and variables, where m is the number of items and T is the number of time periods. To our knowledge this is the first polynomial algorithm for a multi-item lot-sizing problem with joint set-up costs in which a batch may include any mix of items. Our result complements a recent result of Levi et al. [10] showing that the problem is $\mathcal{NP}$-hard when the storage costs are non-speculative, but the dominance condition above does not hold. We also briefly describe a similar tight and compact linear program for the problem with backlogging under more restrictive storage and backlogging cost conditions and when the batch size is arbitrarily large. The full proof and details of the backlogging case can be found in Anily et al. [3].

In general, dynamic lot-sizing problems with capacity restrictions are known to be hard problems. When only one batch is allowed in each period and the capacity limitation is time-dependent, even the single item case is NP-Hard, see Florian et al. [8] and Bitran and Yanasse [4]. When the capacity limitations are time independent, the single item problem is solvable in polynomial time, see Florian and Klein [7] and van Hoesel and Wagelmans [17]. Pochet and Wolsey [14] considered the single item problem with multiple batches, where the set-up, inventory holding and unit production costs are time-dependent. They designed an $O(T^3)$ algorithm, which is based on finding a shortest path in an appropriately defined network. Lee [9] addressed the single item multiple batch problem in which there exists a setup cost for ordering in a particular period, in addition to a different setup cost incurred for each batch, and presented an $O(T^4)$ procedure to solve it.

The special case of the single item problem with non-speculative costs, also called Wagner–Whitin costs, arises very frequently and has received special treatment. In the uncapacitated case, Wagelmans et al. [19] and Federgruen and Tzur [6] have shown that there is an $O(T)$ algorithm, and Pochet and Wolsey [15] have derived a tight and compact linear programming description in both the uncapacitated and constant capacity cases.

The effectiveness of an MIP approach based on such tight linear programs, or on valid inequality descriptions of the convex hulls of solutions, has been demonstrated on various lot-sizing problems starting with Eppen and Martin [5]. The recent book of Pochet and Wolsey [16] classifies and presents the state-of-the-art on the formulation and solution of a wide variety of production planning problems by mixed integer programming, and demonstrates the effectiveness of tight extended formulations on several industrial cases.

Results concerning polynomial algorithms for multi-item problems are limited. Exceptions are the multi-item discrete lot-sizing problems with and without backlogging in which a limited number of items are produced per period, see Miller and Wolsey [12]. The problem studied here with multiple items and multiple batches has been studied recently. Anily and Tzur [1] developed a $O(mT^{m+5})$ dynamic programming algorithm, assuming production, holding and batch/set-up costs to be constant over time. Note that this algorithm has polynomial running time for a fixed number of items, but in practice its running time is prohibitive even for very small values of m. Finally Anily and Tzur [2] contains an optimal search algorithm and heuristics to solve the problem.

In Sect. 2 we formally present the problem and the assumptions on the storage costs, and then develop the results and proofs. In Sect. 3 we state the results for the backlogging case. In Sect. 4 we present some computational results indicating the potential value of these reformulations in solving joint set-up problems as mixed integer programs, both for cases in which the storage cost conditions hold and also when they are violated. We conclude with some remarks concerning possible extensions and open questions.

2 The multi-item joint set-up problem

First we present the notation needed to describe the multi-item problem.
T is the number of periods.
m is the number of items.
d_t^i is the demand of item i in period t for $1 \le i \le m$, $1 \le t \le T$.
h_t^i is the unit storage cost of item i in period t for $1 \le i \le m$, $1 \le t \le T$.

q_t is the fixed cost for producing a full or partial batch in period t.
C is the batch capacity.
v_t is the maximum number of batches allowed in period t. Note that traditionally in lot-sizing models one takes $\mathbf{v} = (v_1, \dots, v_T) = (1, \dots, 1)$ as one allows only one set-up (or a single batch) per period.

To formulate the problem as a mixed integer program, we introduce the following decision variables:
y_t—the number of batches produced in period t.
x_t^i—the production quantity of item i in period t.
s_t^i—the storage quantity of item i at the end of period t, with $s_0^i = 0$ for all i.
The problem, denoted FAM (for a family set-up), can be formulated as the following mixed integer linear program:

$$\min \sum_{i=1}^{m} \sum_{t=1}^{T} h_t^i s_t^i + \sum_{t=1}^{T} q_t y_t \tag{1}$$

$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \text{for all } i, t \tag{2}$$

$$\sum_{i=1}^{m} x_t^i \le C y_t \quad \text{for all } t \tag{3}$$

$$s_0^i = 0 \quad \text{for all } i, \mathbf{y} \le \mathbf{v} \tag{4}$$

$$\mathbf{x} \in \mathbb{R}_+^{mT}, \mathbf{s} \in \mathbb{R}_+^{m(T+1)}, \ \mathbf{y} \in \mathbb{Z}_+^T \tag{5}$$

Recall the nonspeculative and dominance conditions on the storage costs:
$h_t^1 \ge h_t^2 \ge \dots \ge h_t^m \ge h_t^{m+1} \equiv 0$ for all t.
With these additional conditions the problem class is denoted FAM^*.

We introduce the idea of surrogate products, where the i-th surrogate product is composed of the i items having the largest storage costs. (Any two items with identical storage costs can be combined). For this surrogate product, we introduce the following surrogate variables: $S_t^i = \sum_{j=1}^i s_t^j$ are the surrogate storage variables, $X_t^i = \sum_{j=1}^i x_t^j$ are the surrogate production variables, $D_t^i = \sum_{j=1}^i d_t^j$ are the surrogate demands and $H_t^i = h_t^i - h_t^{i+1} \ge 0$ the surrogate storage costs. Note that $\sum_{i=1}^m \sum_{t=1}^T h_t^i s_t^i = \sum_{i=1}^m \sum_{t=1}^T h_t^i (S_t^i - S_t^{i-1}) = \sum_{i=1}^m \sum_{t=1}^T (h_t^i - h_t^{i+1}) S_t^i = \sum_{i=1}^m \sum_{t=1}^T H_t^i S_t^i$, and $X_t^i \le C y_t$ for all i, t, so we obtain an equivalent formulation:

$$\min \sum_{i=1}^{m} \sum_{t=1}^{T} H_t^i S_t^i + \sum_{t=1}^{T} q_t y_t \tag{6}$$

$$S_{t-1}^i + X_t^i = D_t^i + S_t^i \quad \text{for all } i, t \tag{7}$$

$$X_t^i \le C y_t \quad \text{for all } i, t \tag{8}$$

$$S_t^i \ge S_t^{i-1}, X_t^i \ge X_t^{i-1} \quad \text{for all } i, t \tag{9}$$

$$S_0^i = 0 \quad \text{for all } i, \mathbf{y} \le \mathbf{v} \tag{10}$$

$$\mathbf{X} \in \mathbb{R}_+^{mT}, \mathbf{S} \in \mathbb{R}_+^{m(T+1)}, \ \mathbf{y} \in \mathbb{Z}_+^T \tag{11}$$

where constraints (7) are obtained by summing constraints (2) and constraints (9) come from the non-negativity of s_t^i and x_t^i, respectively. Note that there is a bijection between feasible $(\mathbf{x}, \mathbf{s}, \mathbf{y})$ solutions of (2)–(5) and feasible $(\mathbf{X}, \mathbf{S}, \mathbf{y})$ solutions of (7)–(11).

Our first relaxation is precisely to drop the constraints (9) giving:

$$\min \sum_{i=1}^{m} \sum_{t=1}^{T} H_t^i S_t^i + \sum_{t=1}^{T} q_t y_t \tag{12}$$

$$S_{t-1}^i + X_t^i = D_t^i + S_t^i \quad \text{for all } i, t \tag{13}$$

$$X_t^i \leq C y_t \quad \text{for all } i, t \tag{14}$$

$$S_0^i = 0 \quad \text{for all } i, \mathbf{y} \leq \mathbf{v} \tag{15}$$

$$\mathbf{X} \in \mathbb{R}_+^{mT}, \mathbf{S} \in \mathbb{R}_+^{m(T+1)}, \ \mathbf{y} \in \mathbb{Z}_+^T \tag{16}$$

Letting $D_{tl}^i \equiv \sum_{u=t}^{l} D_u^i$, and aggregating the balance constraints (13) and the capacity constraints (14), we obtain a second relaxation:

$$\min \sum_{i=1}^{m} \sum_{t=1}^{T} H_t^i S_t^i + \sum_{t=1}^{T} q_t y_t \tag{17}$$

$$S_{t-1}^i + C \sum_{u=t}^{l} y_u \geq D_{tl}^i \quad \text{for } 1 \leq t \leq l \leq T, \ i = 1, \ldots, m \tag{18}$$

$$S_0^i = 0 \quad \text{for all } i, \mathbf{y} \leq \mathbf{v} \tag{19}$$

$$\mathbf{S} \in \mathbb{R}_+^{m(T+1)}, \ \mathbf{y} \in \mathbb{Z}_+^T, \tag{20}$$

where constraints (18) state that for each surrogate item i, the sum of the initial inventory in period t and the capacity in periods $t, \ldots, \ell$ is at least as large as the demand of the item in these periods. Let X^{FAM^*} be the feasible region (18)–(20).

Note first that the storage cost conditions on h_t^i translate into the condition $H_t^i \geq 0$ for all i, t.

The observations below are almost immediate because the surrogate stock variables S_t^i are unbounded from above, and thus in an extreme point solution of $\text{conv}(X^{FAM^*})$ at least one of the inequalities (18) must be satisfied at equality for each pair i, t.

Observation 1 *If $H_t^i < 0$ for some $i \in \{1, \ldots, m\}, t \in \{1, \ldots, T\}$, then*

$$\min\{\mathbf{HS} + \mathbf{qy} : (\mathbf{S}, \mathbf{y}) \in X^{FAM^*}\} \to -\infty.$$

Observation 2 *Every extreme point of* $\text{conv}(X^{FAM^*})$ *is of the form* $(\mathbf{S}^*, \mathbf{y}^*)$ *where* $\mathbf{y}^* \in \{\mathbf{y} \in [\mathbf{0}, \mathbf{v}] \cap \mathbb{Z}^T : \sum_{u=1}^{t} y_u \geq \lceil \frac{D_{1t}^m}{C} \rceil$ *for* $1 \leq t \leq T\}$ *and* $S_{t-1}^{*i} = \max_{k=t,\ldots,T} (D_{tk}^i - C \sum_{u=t}^{k} y_u^*)^+$ *for* $1 \leq t \leq T, \ 1 \leq i \leq m$.

Lemma 1 *In every extreme point of* $\operatorname{conv}(X^{FAM^*})$

$$0 \le D_t^i + S_t^i - S_{t-1}^i = \min[D_t^i + S_t^i, Cy_t] \le Cy_t \quad \text{for } 1 \le t \le T, 1 \le i \le m.$$

Proof Consider an extreme point $(\mathbf{S}, \mathbf{y})$ of $\operatorname{conv}(X^{FAM^*})$. According to Observation 2 this extreme point satisfies the following inequalities: $S_{t-1}^i \ge 0$ and $S_{t-1}^i \ge (D_{t\tau}^i - C\sum_{u=t}^{\tau} y_u)^+$ for $1 \le t \le \tau \le T$, where at least one inequality holds as equality for all $1 \le t \le T,\ 1 \le i \le m$.

We first show that $D_t^i + S_t^i - S_{t-1}^i \le \min[D_t^i + S_t^i, Cy_t]$ by showing that $D_t^i + S_t^i - S_{t-1}^i \le Cy_t$. For this we consider two distinct cases, $S_t^i = 0$ or $S_t^i > 0$. If $S_t^i = 0$, then by Observation 2 $S_{t-1}^i \ge D_t^i - Cy_t$ and therefore $D_t^i + S_t^i - S_{t-1}^i = D_t^i - S_{t-1}^i \le D_t^i - (D_t^i - Cy_t) = Cy_t$. If $S_t^i > 0$, then from Observation 2 there exists τ, $t+1 \le \tau \le T$, such that $S_t^i = D_{t+1,\tau}^i - C\sum_{u=t+1}^{\tau} y_u$. Observation 2 also implies that $S_{t-1}^i \ge D_{t,\tau}^i - C\sum_{u=t}^{\tau} y_u$. Therefore, $D_t^i + S_t^i - S_{t-1}^i = D_t^i + D_{t+1,\tau}^i - C\sum_{u=t+1}^{\tau} y_u - S_{t-1}^i \le D_t^i + D_{t+1,\tau}^i - C\sum_{u=t+1}^{\tau} y_u - D_{t,\tau}^i + C\sum_{u=t}^{\tau} y_u = Cy_t$.

To complete the proof, we show that either $D_t^i + S_t^i - S_{t-1}^i \ge D_t^i + S_t^i$ or $D_t^i + S_t^i - S_{t-1}^i \ge Cy_t$. The former inequality is clearly satisfied when $S_{t-1}^i = 0$. If $S_{t-1}^i > 0$, there are two cases. If $S_{t-1}^i = D_t^i - Cy_t$, then $D_t^i + S_t^i - S_{t-1}^i = D_t^i + S_t^i - (D_t^i - Cy_t) = S_t^i + Cy_t \ge Cy_t$. If $S_{t-1}^i = D_{t\tau}^i - C\sum_{u=t}^{\tau} y_u$ with $\tau > t$, then using the fact that $S_t^i \ge D_{t+1,\tau}^i - C\sum_{u=t+1}^{\tau} y_u$, we get that $D_t^i + S_t^i - S_{t-1}^i \ge D_t^i + D_{t+1\tau}^i - C\sum_{u=t+1}^{\tau} y_u - (D_{t\tau}^i - C\sum_{u=t}^{\tau} y_u) = Cy_t$. □

Now we can state and prove the two principal results of this paper.

Theorem 2 *The relaxation* $\min\{\mathbf{HS} + \mathbf{qy} : (\mathbf{S}, \mathbf{y}) \in X^{FAM^*}\}$ *solves the multi-item lot-sizing problem* (1)–(5) *if and only if* $\mathbf{H} \ge 0$.

Proof We have shown above that the costs are unchanged under the change of the surrogate variables. When $H_t^i \ge 0$ for all i and t, the problem $\min\{\mathbf{HS} + \mathbf{qy} : (\mathbf{S}, \mathbf{y}) \in X^{FAM^*}\}$ is a relaxation of the multi-item lot-sizing problem, and we only need to show that if $(\mathbf{S}, \mathbf{y})$ is an optimal (extreme point) solution of this relaxation, then the corresponding solution $(\mathbf{x}, \mathbf{s}, \mathbf{y})$ with $s_t^i = S_t^i - S_t^{i-1}$, $X_t^i = D_t^i + S_t^i - S_{t-1}^i$ and $x_t^i = X_t^i - X_t^{i-1}$ solves the original problem. Given $\mathbf{y} \in \mathbb{Z}_+^T$ which is optimal for $\min\{\mathbf{HS} + \mathbf{qy} : (\mathbf{S}, \mathbf{y}) \in X^{FAM^*}\}$, (4) holds by definition. Moreover, there exists a corresponding optimal solution $\mathbf{S}$ with $S_{t-1}^i = \max_{l \ge t}(D_{tl}^i - C\sum_{u=t}^{l} y_u)^+$ for all i exactly as in Observation 2. Now as $D_t^i \le D_t^{i+1}$, we have that $S_{t-1}^i \le S_{t-1}^{i+1}$, and thus $s_{t-1}^i \ge 0$. Then $S_t^i \le S_t^{i+1}$, $D_t^i \le D_t^{i+1}$ and from Lemma 1 $X_t^i = \min[D_t^i + S_t^i, Cy_t]$. It follows that $X_t^i \le X_t^{i+1}$, and so $x_t^i \ge 0$. Also as $(\mathbf{S}, \mathbf{X})$ satisfy the flow conservation equations (13) by definition, $(\mathbf{s}, \mathbf{x})$ satisfy the flow conservation constraints (2). Finally, $X_t^m \le Cy_t$, so $\sum_{i=1}^{m} x_t^i \le Cy_t$ and (3) is satisfied. Thus $(\mathbf{x}, \mathbf{s}, \mathbf{y})$ is feasible for the original problem (1)–(5). □

The next result provides a compact linear programming formulation for $\operatorname{conv}(X^{FAM^*})$.

Theorem 3 conv(X^{FAM^*}) =*proj*$_{\mathbf{s},\mathbf{y}} Q^{FAM^*}$ *where the polyhedron* Q^{FAM^*} *is described by the following constraints:*

$$S^i_{t-1} = C\mu^i_t + C\sum_{u=t}^{T} f^i_{tu}\delta^i_{tu} \quad \text{for all } i, t \tag{21}$$

$$\sum_{u=t}^{l} y_u + \mu^i_t + \sum_{\{u: f^i_{tu} \geq f^i_{tl}\}} \delta^i_{tu} \geq \lfloor D^i_{tl}/C \rfloor + 1$$
$$\text{for all } i, t, l \text{ with } 1 \leq t \leq l \leq T \tag{22}$$

$$\sum_{u=t}^{T+1} \delta^i_{tu} = 1 \quad \text{for all } i, t \tag{23}$$

$$S^i_0 = 0 \quad \text{for all } i, \mathbf{y} \leq \mathbf{v} \tag{24}$$

$$\mathbf{S} \in \mathbb{R}^{m(T+1)}_+, \mathbf{y} \in \mathbb{R}^T_+, \mu \in \mathbb{R}^{mT}_+, (\delta^i_{tt}, \ldots, \delta^i_{tT}, \delta^i_{t,T+1}) \in \mathbb{R}^{T-t+2}_+$$
$$\text{for all } i, t \tag{25}$$

where $f^i_{tl} = D^i_{tl}/C - \lfloor D^i_{tl}/C \rfloor$ *for all* i, t, l *with* $1 \leq t \leq l \leq T$ *and* $f^i_{t,T+1} = 0$ *for all* i, t.

Proof The motivation for the variables of this new formulation lies in the fact that in an extreme point solution of (18)–(20), S^i_{t-1} mod C must take one of the values $Cf^i_{tt}, \ldots, Cf^i_{tT}, Cf^i_{t,T+1}$. Introducing corresponding 0-1 variables: $\delta^i_{tu} = 1$ if S^i_{t-1} mod $C = Cf^i_{tu}$ and μ^i_t for the integer part, we have that in every extreme point

$$S^i_{t-1} = C\sum_{u=t}^{T} f^i_{tu}\delta^i_{tu} + C\mu^i_t \quad \text{for all } i, t$$

$$\sum_{u=t}^{T+1} \delta^i_{tu} = 1 \quad \text{for all } i, t$$

$$\delta^i_t \in \{0, 1\}, \mu^i_t \in \mathbb{Z}^1_+ \quad \text{for all } i, t.$$

The proof consists of three steps: (i) to show that conv$(X^{FAM^*}) \subseteq$proj$_{\mathbf{s},\mathbf{y}} Q^{FAM^*}$; (ii) to show that proj$_{\mathbf{s},\mathbf{y}} Q^{FAM^*} \cap (\mathbb{R}^{m(T+1)} \times \mathbb{Z}^T) \subseteq X^{FAM^*}$ and (iii) to show that $\mathbf{y}$ is integer-valued in an extreme point of Q^{FAM^*} and thus also in an extreme point of proj$_{\mathbf{s},\mathbf{y}} Q^{FAM^*}$.

(i) If $(\mathbf{s}, \mathbf{y})$ is an extreme point of conv(X^{FAM^*}), there exist δ, μ integer such that $(\mathbf{s}, \mathbf{y}, \delta, \mu)$ satisfies (18)–(20),(21),(23). Substituting for S^i_{t-1} using (21), (18) becomes $\sum_{u=t}^{l} y_u + \sum_{u=t}^{T} f^i_{tu}\delta^i_{tu} + \mu^i_t \geq D^i_{tl}/C$. Then using (23), $\sum_{\{u: f^i_{tu} < f^i_{tl}\}} f^i_{tu}\delta^i_{tu} < f^i_{tl}$ and thus $\sum_{u=t}^{l} y_u + \sum_{\{u: f^i_{tu} \geq f^i_{tl}\}} f^i_{tu}\delta^i_{tu} + \mu^i_t > D^i_{tl}/C - f^i_{tl} = \lfloor D^i_{tl}/C \rfloor$. Now Chvàtal-Gomory rounding shows that (22) is a valid inequality when $\mathbf{y}, \delta, \mu$ are integer. As extreme rays of conv(X^{FAM^*}) are generated by increasing the μ_t^i variables, (22) is valid for conv(X^{FAM^*}).

(ii) Here it suffices to show that if $(\mathbf{s}, \mathbf{y}, \delta, \mu) \in Q^{FAM^*}$ with y integer, then (s, y) satisfies (18). Letting $\alpha = \sum_{u: f_{tu}^i \geq f_{tl}^i} \delta_{tu}^i$ with $0 \leq \alpha \leq 1$, we have that

$$\begin{aligned} S_{t-1}^i + C\sum_{u=t}^{l} y_u &= C\mu_t^i + C\sum_{u=t}^{T} f_{tu}^i \delta_{tu}^i + C\sum_{u=t}^{l} y_u \\ &\geq C(\lfloor \frac{D_{tl}^i}{C} \rfloor + 1 - \sum_{\{u: f_{tu}^i \geq f_{tl}^i\}} \delta_{tu}^i) + C\sum_{u=t}^{T} f_{tu}^i \delta_{tu}^i \\ &\geq C\lfloor \frac{D_{tl}^i}{C} \rfloor + C\{(1-\alpha) + f_{tl}^i \alpha\} \\ &\geq C\lfloor \frac{D_{tl}^i}{C} \rfloor + Cf_{tl}^i = D_{tl}^i. \end{aligned}$$

(iii) We show that the 0-1 matrix A associated with the constraints (22)–(23) is totally unimodular. We use the characterization that for all column subsets J, there exists a partition J^+, J^- of J such that

$$\left| \sum_{u \in J^+} a_{ru} - \sum_{u \in J^-} a_{ru} \right| \leq 1$$

for every row r of the matrix.

Let J_y be the columns associated to y_t variables in J and let J_δ^i be the columns associated to μ^i, δ^i variables in J.

We assign elements of J_y alternately to J^+ and J^- working from y_1 up to y_T.

Now fix t, and let $\tau_t = \min\{u : u \geq t, u \in J_y\}$.

For each i, treat the variables μ_t^i, δ_{tu}^i in the order $(\mu_t^i, \delta_{tu_1}^i, \delta_{tu_2}^i, \ldots, \delta_{tu_{T-t+1}}^i, \delta_{t,T+1}^i)$ where $f_{tu_1}^i \geq f_{tu_2}^i \geq \cdots \geq f_{tu_{T-t+1}}^i \geq f_{tu_{T+1}}^i = 0$. Observe that the corresponding constraint submatrix, defined by the variables in J_δ^i with i, t fixed, and the constraints (22) is a consecutive 1's matrix with 1's in the first column.

Now consider row $r = (i, t, \ell)$ of (22). If $\tau_t \in J^+$, assign the columns of J_δ^i alternately starting with J^-.

We have by construction that as $\tau_t \in J^+$,

$$0 \leq \sum_{u \in J^+ \cap J_y} a_{ru} - \sum_{u \in J^- \cap J_y} a_{ru} \leq 1.$$

Also from the assignment to J_δ^i,

$$-1 \leq \sum_{u \in J^+ \cap J_\delta^i} a_{ru} - \sum_{u \in J^- \cap J_\delta^i} a_{ru} \leq 0.$$

By addition we obtain that

$$-1 \leq \sum_{u \in J^+} a_{ru} - \sum_{u \in J^-} a_{ru} \leq 1.$$

If $\tau_t \in J^-$, the argument is similar. Thus the matrix is TU, and the extreme points of Q^{FAM^*} have $\mathbf{y}$ integer. □

Putting these two theorems together, we have the result that is the basis of our subsequent computations.

Corollary 4 *An optimal extreme point solution of the linear program* $\min\{\mathbf{HS} + \mathbf{qy} : (\mathbf{S}, \mathbf{y}, \mu, \delta) \in Q^{FAM^*})$ *solves* FAM^**. It suffices to set* $s_t^i = S_t^i - S_t^{i-1}$ *and* $x_t^i = d_t^i + s_t^i - s_{t-1}^i$ *for all* i, t *to obtain an optimal solution to* FAM^**.*

The following observation is also useful in practice. When $m = 1$, X^{FAM^*} reduces to the set $X^{WW-CC} = \{(\mathbf{s}, \mathbf{y}) \in \mathbb{R}_+^n \times \mathbb{Z}_+^n : s_{k-1} + C \sum_{u=k}^{t} y_u \geq d_{kt}\ 1 \leq k \leq t \leq T, \mathbf{y} \leq \mathbf{v}\}$, arising from the problem with Wagner–Whitin costs and constant capacities, that has been studied earlier. In particular, Q^{FAM^*} reduces to one of the known extended formulations for $\mathrm{conv}(X^{WW-CC})$, and a "mixing inequality" description of $\mathrm{conv}(X^{WW-CC})$ in the original variables with an $O(T^2 \log T)$ separation algorithm is known, see in Pochet and Wolsey [15,16]. Theorem 3 is closely related to a similar result for mixing sets with common integer variables in Miller and Wolsey [13] and can also be written as

$$\mathrm{conv}(X^{FAM^*}(\mathbf{S}, \mathbf{y})) = \bigcap_{i=1}^{m} \mathrm{conv}(X^{WW-CC}(\mathbf{S^i}, \mathbf{y})).$$

Corollary 5 *An inequality description of* $\mathrm{conv}(X^{FAM^*}(\mathbf{S}, \mathbf{y}))$ *is known, and there is an* $O(mT^2 \log T)$ *separation algorithm.*

When C is very large, and in particular $C \geq \sum_{i=1}^{m} \sum_{u=1}^{T} d_u^i = D_{1T}^m$, the problem becomes uncapacitated and is denoted by $FAM^* - U$. Using the explicit description of the convex hull of the single item uncapacitated problem from Pochet and Wolsey [15], we get:

Corollary 6 *The linear program*

$$\min \sum_{i=1}^{m} \sum_{t=1}^{T} H_t^i S_t^i + \sum_{t=1}^{T} q_t y_t \tag{26}$$

$$S_{t-1}^i + \sum_{u=t}^{l} D_{ul}^i y_u \geq D_{tl}^i \quad \text{for } 1 \leq t \leq l \leq T,\ i = 1, \ldots, m \tag{27}$$

$$S_0^i = 0 \quad \text{for all } i, \quad y_t \leq 1 \quad \text{for all } t \tag{28}$$

$$s_t^i = S_t^i - S_t^{i-1} \quad \text{for all } i, t \tag{29}$$

$$x_t^i = s_t^i - s_{t-1}^i + d_t^i \quad \text{for all } i, t \tag{30}$$
$$\mathbf{S} \in \mathbb{R}_+^{m(T+1)}, \mathbf{y} \in \mathbb{R}_+^T \tag{31}$$

solves the uncapacitated problem $FAM^* - U$.

3 The backlogging case: Uncapacitated

In this section we consider the backlogging version of the joint set-up problem, that we denote by $FAM - B$. It can be formulated as the following mixed integer program:

$$z = \min \sum_{i=1}^{m}\sum_{t=1}^{T} h_t^i s_t^i + \sum_{i=1}^{m}\sum_{t=1}^{T} b_t^i r_t^i + \sum_{t=1}^{T} q_t y_t$$
$$s_{t-1}^i - r_{t-1}^i + x_t^i = d_t^i + s_t^i - r_t^i \quad \text{for all } i, t$$
$$\sum_{i=1}^{m} x_t^i \le C y_t \quad \text{for all } t$$
$$s_0^i = r_0^i = 0 \quad \text{for all } i,$$
$$\mathbf{x} \in \mathbb{R}_+^{mT},\ \mathbf{s}, \mathbf{r} \in \mathbb{R}_+^{m(T+1)},\ \mathbf{y} \in \mathbb{Z}_+^T,$$

where r_t^i is a variable denoting the backlog size of item i in period t, and b_t^i is the backlogging cost per unit of item i backlogged in period t.

It is now natural to ask under what conditions, if any, a relaxation based on m surrogate items solves $FAM - B$. We show that it can be done for the uncapacitated version of this problem, denoted $FAM - B - U$, in which the capacity C is replaced by a large parameter M s.t. $M \ge D_{1T}^m = \sum_{i=1}^m \sum_{t=1}^T d_t^i$.

Assuming an ordering of the items, we proceed as before and construct the same surrogate items using in addition $R_t^i = \sum_{j=1}^i r_t^j$, $B_t^i = b_t^i - b_t^{i+1}$, and as before we obtain a relaxation

$$\min \sum_{i=1}^{m}\sum_{t=1}^{T} H_t^i S_t^i + \sum_{i=1}^{m}\sum_{t=1}^{T} B_t^i R_t^i + \sum_{t=1}^{T} q_t y_t \tag{32}$$
$$S_{t-1}^i + M\sum_{u=t}^{l} y_u + R_l^i \ge D_{tl}^i \quad \text{for } 1 \le t \le l \le T,\ i = 1, \ldots, m, \tag{33}$$
$$S_0^i = R_0^i = 0 \quad \text{for all } i, \tag{34}$$
$$\mathbf{S}, \mathbf{R} \in \mathbb{R}_+^{m(T+1)},\ \mathbf{y} \in \mathbb{Z}_+^T, \tag{35}$$

whose feasible region (33)–(35) is denoted X^{FAM^*-B-U}. When $m = 1$, this reduces to the set $X^{WW-U-B} = \{(\mathbf{s}, \mathbf{r}, \mathbf{y}) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{Z}_+^n : s_{k-1} + M\sum_{u=k}^t y_u + r_t \ge d_{kt}\ 1 \le k \le t \le T\}$, arising from the uncapacitated problem with Wagner–Whitin costs ($h_t, b_t \geq 0$ for all t), and backlogging for which the convex hull is known. Specifically we can write

$$X^{FAM^*-B-U}(\mathbf{S}, \mathbf{R}, \mathbf{y}) = \bigcap_{i=1}^{m} X^{WW-U-B}(\mathbf{S^i}, \mathbf{R^i}, \mathbf{y}).$$

Below we examine the strength of this relaxation and examine whether its convex hull can be described. First we introduce the assumptions on the storage and backlog costs that we will use:

(i) $h_t^i = h^i, b_t^i = b^i$ for all i, t

(ii) $h^i - h^{i+1} \geq 0, b^i - b^{i+1} \geq 0$ for all i

(iii) $b^i = \kappa h^i$ for all i.

With these additional conditions, the problem is denoted $FAM^* - B - U$. With time invariance, conditions (i) and (ii) are natural extensions from the case without backlogging. Condition (iii) then ensures that between any two production periods, those periods in which demand is satisfied from stock and those satisfied by backlogging are the same for each item.

Theorem 7 *The relaxation* $\min\{\mathbf{HS} + \mathbf{BR} + \mathbf{qy} : (\mathbf{S}, \mathbf{R}, \mathbf{y}) \in X^{FAM^*-B-U}\}$ *solves* $FAM^* - B - U$.

Theorem 8

$$\mathrm{conv}(X^{FAM^*-B-U}(\mathbf{S}, \mathbf{R}, \mathbf{y})) = \bigcap_{i=1}^{m} \mathrm{conv}(X^{WW-U-B}(\mathbf{S^i}, \mathbf{R^i}, \mathbf{y})).$$

The proof of these two theorems can be found in Anily et al. [3]. That of Theorem 7 is similar to that of Theorem 2, and that of Theorem 8 is a simple generalization of the proof for the case $m = 1$ as it appears in Pochet and Wolsey [15]. The latter article provides a compact extended formulation and a fast separation algorithm for $\mathrm{conv}(X^{WW-U-B}$. Alternatively one can use the facility location formulation for uncapacitated lot-sizing with backlogging that is also known to be tight, see, for example, Levi et al. [11].

The storage and backlog cost conditions (i)–(iii) are very restrictive. However there appears to be little chance of relaxing them significantly. The already strong conditions (i) and (ii) are insufficient. Specifically the relaxation (32)–(35) does not solve the instance with $m = 2, T = 8$, $d^1 = (10, 5, 8, 4, 1, 16, 38, 31)$, $d^2 = (2, 3, 4, 9, 2, 13, 21, 25)$, $h^1 = 0.2, h^2 = 0.1, b^1 = 2.0, b^2 = 0.2, q = 101$.

It has also been pointed out by a referee that the problem $FAM^* - B - U$ is easily solved by dynamic programming.

4 Computation

Here our aim is to indicate the potential of the extended formulations developed in the previous sections in solving the multi-item lot-sizing problem with joint set-up costs FAM. Much effort has been spent in recent years in using polyhedral combinatorics to develop special purpose branch-and-cut algorithms, and our results could be used in this way. However, extended formulations can be used directly with a standard mixed integer programming solver, and do not require the development of special separation routine or branch-and-cut software.

It is also important to observe that the extended formulations are valid for relaxations of FAM, and can therefore be used whether or not the Wagner–Whitin and/or Dominance conditions hold. It is thus natural to ask to what extent the formulation is effective even when these conditions are violated. Note however that many models encountered in practice have storage costs per item that are constant throughout the time horizon, so the storage cost conditions are very often satisfied.

Our computational tests are limited to (i) showing how a standard MIP solver performs on one set of randomly generated instances of FAM^* when given the original formulation, or one tightened by either the uncapacitated or constant capacity extended formulations, and to (ii) indicating that the extended formulations can be used and can be effective on instances of FAM for which the storage cost conditions are not satisfied.

4.1 Instances of FAM^* satisfying the cost conditions

For the joint set-up problem FAM^*, we consider instances with $m = 30$ and $T = 50$, and three capacity levels $C \in \{50, 120, 250\}$. The instances are generated randomly as follows:
$d_t^i = \lceil 5*rand \rfloor$ ($\lceil x \rfloor$ denotes the closest integer to x) for all i, t, $h_t^m = 0.05+0.1*rand$ for all t and $h_t^i = h_t^{i+1} + 0.05*rand$ for all i, t with $i < m$, and $q_t = \lceil 75+50*rand \rfloor$ for all t, where $rand$ is generated each time uniformly in [0, 1].
The bounds v_t for $t = 1 \dots T$ are set to 100 and are inactive. The instances with $C = 50$ can be considered as tightly constrained possibly requiring multiple batches per period, whereas those with $C = 250$ are relatively uncapacitated problems in the sense that typically $y_t \in \{0, 1\}$, and the bound is inactive.

For each instance we compare three options. The first is to solve the MIP (1)–(5), denoted "O" (original). The second is to add the linear program (27)–(31) based on the uncapacitated problem to the formulation (1)–(5) with S_t^i replaced by $\sum_{j=1}^i s_t^j$, denoted "U" (uncapacitated). The third is to add the linear program (21)–(26) based on the constant capacity problem to the formulation (1)–(5), again with S_t^i replaced by $\sum_{j=1}^i s_t^j$, denoted "CC" (constant capacity). All three MIPs are then solved with the Xpress-MP system, Version 16.01.01, using the standard defaults running under Windows XP on a 1.6 GHz IBM Thinkpad.

In the two Tables below we will use the following notation: LP denotes the value of the linear programming relaxation of the (re)formulation, XLP the value after the further addition of Xpress-MP cutting planes at the top node, IP the optimal value, and BLB and BIP the best lower and upper bounds on termination.

Table 1 Average of 5 FAM^* instances with 30 items and 50 periods

C		r	c	LP/IP	XLP/IP	BLB/IP	BIB/IP	S	Nodes
50	O	1,550	3,050	97.6	98.5	99.0	100.1	300	48,300
50	U	39,697	3,050	97.8	98.5	98.9	100.2	300	7,500
50	CC	42,800	42,800	100	100	100	100	11	1
120	O	1,550	3,050	81.5	94.2	97.1	100.3	300	17,700
120	U	39,697	3,050	95.8	96.6	98.3	100.4	300	3,700
120	CC	42,800	42,800	100	100	100	100	34	1
250	O	1,550	3,050	49.3	84.4	94.0	100.3	300	11,700
250	U	39,697	3,050	100.0	100.0	100.0	100.0	9	3
250	CC	42,800	42,800	100	100	100	100	45	1

Results for the FAM^* instances are presented in Table 1. In the first column we indicate the value of the capacity C, in the second column we indicate which reformulation is used: O, U, or CC. In the next two columns we specify the number of rows r and columns c of the problem after reformulation. The next four columns present the average values of the ratios LP/IP,XLP/IP,BLB/IP and BIP/IP taken over 5 instances, written as percentages. The last two columns give the average total time in seconds and the average number of nodes in the tree. The maximum run time was set to be 300 seconds. Note that the instances have just 50 integer variables.

We now comment briefly on the results in Table 1. As a consequence of Theorems 2 and 3, the constant capacity reformulation allows us to solve all the instances by linear programming. The fact that when $C = 250$ the uncapacitated reformulation requires only three nodes on average confirms that these instances are relatively uncapacitated. With the original formulation, we observe that the duality gaps are much smaller for the capacitated cases with $C \in \{120, 50\}$ than for the uncapacitated case with $C = 250$. One also sees from column BIP/IP that the quality of the best feasible solution found after 300 seconds is always remarkably good, especially with the original formulation.

4.2 Instances of FAM violating the cost conditions

Here we consider instances of FAM satisfying neither the Wagner–Whitin nor the Dominance conditions. Specifically the instances are generated as before, except that $h_t^m = -0.15 + rand$, where $rand$ is generated each time uniformly in $[0, 1]$, so there is a 15% chance on average of h_t^m being negative. As before, $h_t^i = h_t^{i+1} + 0.05 * rand$ for all i, t with $i < m$. Again we generated five instances for which we consider three different capacity levels $C \in \{50, 120, 250\}$. Each instance is solved first with the initial formulation (1)–(5), denoted "O" as above, and then with a tightened formulation.

Two observations motivate our choice of reformulation for these instances.

1. As these instances do not satisfy the cost conditions, every subset of items is a potential surrogate item. Our choice here is to use the m singleton sets (single items), and m surrogates of the form $\{1, \ldots, i\}$ where the items are ordered by

Table 2 Average of 5 FAM instances with 30 items and 50 periods

C		r	c	LP/IP	XLP/IP	BLB/IP	BIB/IP	S	Nodes	Opt
50	O	1,550	3,050	98.0	98.7	99.1	100.2	300	40,509	0
50	CC	5,3150	51,650	99.4	99.8	99.9	100.1	154	494	4
120	O	1,550	3,050	85.2	95.2	97.8	100.2	300	18,660	0
120	CC	45,500	44,000	99.2	99.3	99.8	100.0	247	140	3
250	O	1,550	3,050	54.3	82.8	90.8	100.9	300	10,732	0
250	U	37,955	3,050	99.6	99.6	100	100	22	42	5

nonincreasing average storage cost $\bar{h}_i = \sum_t h_t^i$.

2. The complete reformulation using (21)–(26) has a very large number of constraints and variables. However, as shown in Van Vyve and Wolsey [18], approximate versions of these reformulations that are significantly smaller often give bounds almost as good as those provided by the complete formulation. Specifically for the constant capacity reformulation, we choose a parameter TK with $0 \leq TK \leq T-1$. The approximate formulation obtained from (21)–(26) by only introducing the variables δ_{tl}^i and the constraints (22) for values of t, l with $l - t \leq TK$, denoted $Q_{TK}^{FAM^*}$, is a valid relaxation. A similar approximation is obtained in the uncapacitated case if one just generates the constraints (27) for values of t, l with $l - t \leq TK$.

We can now describe the reformulations used in Table 2. For the instances with $C = 50$, we use the formulation (1)–(5), and then add the approximate formulation $Q_{TK}^{FAM^*}$ for each single item set with $TK = 10$ and the approximate formulation $Q_{TK}^{FAM^*}$ for the m surrogate items with $TK = 30$. For $C = 120$, we adopted the same reformulation but with values of $TK = 10$ and 20, respectively. These two reformulations are denoted "CC". For $C = 250$, we added the approximate uncapacitated formulation because it has no additional variables. We used the values $TK = 10$ for the single items and $TK = 20$ for the surrogate items. This formulation is denoted "U".

Table 2 has the same structure as Table 1, except for the additional column in which we indicate how many of the instances are solved to optimality within the time limit of 300 s.

These results suggest that even when the storage cost conditions are not satisfied, the reformulations aid significantly in strengthening the lower bounds and in proving optimality.

5 Further remarks and observations

In the brief computational section we have attempted to indicate one of the simplest ways in which our results can be used in practice, namely just passing a modified MIP formulation to a standard MIP solver. One possible alternative is to work in the original space of variables and add violated inequalities describing the convex hull as cuts. Again the description of the facet-defining inequalities and an efficient separation algorithm are known—based on Observation 4 it suffices to use the separation algorithm for conv(X^{WW-CC}) described in Pochet and Wolsey [16] for each surrogate item. So this approach requires the implementation of an $O(mT^2 \log T)$ separation routine integrated with the MIP solver. Another classical alternative would be to use Lagrangean relaxation or column generation, in which case the subproblem to be solved at each iteration is to optimize over X^{FAM^*}. This leads us to an intriguing open question.

Is there a polynomial dynamic program or a polynomial combinatorial algorithm for FAM^* faster than the linear programming approach derived here?

The so-called Joint Replenishment Problem involving both joint and individual item fixed costs is a more complicated problem whose formulation might also benefit from the results of this paper. Levi et al. [11] give a primal-dual 2-approximation algorithm for the version with backlogging and a very general storage cost structure that is based on the facility location formulation for uncapacitated lot-sizing. More generally our reformulations can be applied to any lot-sizing problem with joint fixed costs.

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