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Portfolio Optimization Using a Block Structure for the Covariance Matrix

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Abstract: Implementing in practice the classical mean-variance theory for portfolio selection often results in obtaining portfolios with large short sale positions. Also, recent papers show that, due to estimation errors, existing and rather advanced mean-variance theory-based portfolio strategies do not consistently outperform the naïve $1/N$ portfolio that invests equally across N risky assets. In this paper, we introduce a portfolio strategy that generates a portfolio, with no short sale positions, that can outperform the $1/N$ portfolio. The strategy is investing in a global minimum variance portfolio (GMVP) that is constructed using an easy to calculate block structure for the covariance matrix of asset returns. Using this new block structure, the weights of the stocks in the GMVP can be found analytically, and as long as simple and directly computable conditions are met, these weights are positive.

Keywords: portfolio optimization, short sale constraints, block covariance matrix, the $1/N$ portfolio

1. INTRODUCTION

According to the seminal work of Markowitz (1952 and 1959) an investor who cares only about the mean and variance of portfolio returns should hold a portfolio on the efficient frontier. In practice, applying the Markowitz mean-variance theory involves estimating the means and covariances of asset returns, and often results in portfolios with large short sale positions. This is true both when the means and covariances are estimated by the traditional sample mean vector and the sample covariance matrix respectively, as well as by more advanced estimation techniques.[1]

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1 See, for example, Ledoit and Wolf (2003), Jagannathan and Ma (2003), DeMiguel et al. (2009a) and Disatnik and Benninga (2007).

Obtaining portfolios with short sale positions can be considered a drawback, since short selling is often restricted by regulators, investment policies of mutual funds sometimes prohibit taking short positions, and many individual investors find short selling onerous or impossible.[2] To the extent that short sales are indeed considered an undesirable feature of portfolio optimization, it is of interest to find ways to produce efficient portfolios with long-only positions (henceforth — long-only portfolios).

Specifically, there is an interest in obtaining a long-only global minimum variance portfolio (henceforth — GMVP), which is, in the mean-variance framework, the portfolio on the efficient frontier with the smallest return variance.[3] The interest in the GMVP stems from the fact that several empirical studies show that in practice out-of-sample (ex-post) it performs at least as well as other frontier portfolios, even when performance is evaluated based on measures that relate to both the return mean and variance (rather than just the variance), such as the ex-post Sharpe ratio.[4] The common explanation for the relatively good ex-post performance of the GMVP is that the derivation of the GMVP requires estimating only the covariance matrix of asset returns, whereas for other efficient portfolios we have to estimate the means of asset returns as well, and that significantly adds to the estimation error.[5]

One way to obtain a long-only GMVP (as well as other long-only frontier portfolios) is to impose on the optimization problem short sale constraints that prevent the portfolio weights from being negative (constrained optimization). However, at least from a theoretical point of view, this procedure is problematic, as it generates a portfolio with weights that can only be found numerically and not analytically.[6] Another problem with imposing the short sale constraints, as noted by Black and Litterman (1992), is that they generate 'corner' solutions with zero weights in many assets.

In this paper, we introduce a new structure for the covariance matrix of asset returns, the *block structure*, which under simple and directly computable conditions generates (in an unconstrained optimization) a long-only GMVP with weights that can be found analytically. These conditions do not necessarily impose 'corner' solutions with many zero weights.

To construct a block covariance matrix, one divides the portfolio's stocks into several groups (blocks). Within each block, the covariance between stocks is identical for all pairs of stocks in the block. The covariance between stocks from different blocks is also identical for all pairs. Thus, in the block structure, the number of covariances associated with each stock is reduced to two: the covariance with the other stocks in the same block (the within-block covariance) and the covariance with the stocks from the other blocks (the between-block covariance).

2 See, for example, regulation SHO of the US Securities and Exchange Commission (SEC), http://www.sec.gov/spotlight/shortsales.htm. Almazan et al. (2004) report that over the 1994–2000 period 69% of their sample of US domestic equity mutual funds were not allowed to short.

3 Note that in practice the GMVP often includes less extreme short sale positions than other efficient portfolios (see, for example, Jagannathan and Ma, 2003). Nevertheless, as we confirm later in this paper, these short sale positions are still quite significant (see also, for example, Disatnik and Benninga, 2007).

4 See, for example, Jorion (1985, 1986 and 1991), Jagannathan and Ma (2003) and DeMiguel et al. (2009a).

5 Jagannathan and Ma (2003), for instance, note that 'the estimation error in the sample mean is so large that nothing much is lost in ignoring the mean altogether when no further information about the population mean is available.'

6 To obtain a solution for the constrained optimization problem, an iterative procedure, based on the Kuhn-Tucker conditions, is commonly used.

We show that the weights of a GMVP constructed using a block covariance matrix can be written as a function of the stock variances and covariances, which goes further than the general solution for the unconstrained problem $\mathbf{w}_{GMVP} = \mathbf{\Sigma}^{-1}\mathbf{1}/\mathbf{1}^{\mathrm{T}}\mathbf{\Sigma}^{-1}\mathbf{1}$, where $\mathbf{\Sigma}^{-1}$ denotes the inverse matrix of the covariance matrix and $\mathbf{1}(\mathbf{1}^{\mathrm{T}})$ denotes a vector (a transpose vector) of ones. In essence, adding the conditions that: (*i*) the variance of each stock is greater than both its within-block covariance and its between-block covariance; and (*ii*) the within-block covariances are not smaller than the between-block covariance is sufficient to ensure obtaining a long-only GMVP.

We find the block structure appealing from an implementation perspective for a combination of reasons. First, it reduces the severe sampling error caused by having to estimate the whole covariance matrix, as in the traditional sample matrix case for instance. That is because, in the block construct, the number of covariances associated with each stock is reduced to two. Second, the condition that the variance of each stock is greater than its two associated covariances implies that we are dealing with relatively small-sized covariances. Thus, like the shrinkage estimators advocated by Ledoit and Wolf (2003, 2004a and 2004b), and the portfolios of estimators advocated by Jagannathan and Ma (2000), Bengtsson and Holst (2002), Disatnik and Benninga (2007) and Fletcher (2009), the block matrix has the appealing property of off-diagonal elements which are shrunk compared to the typically large off-diagonal elements of the traditional sample matrix. Not only are the large off-diagonal elements those responsible for the extreme short positions that are obtained so often when the mean-variance theory is implemented in practice, but as Michaud (1989) states, inverting the sample matrix with its large off-diagonal elements also amplifies the sampling error (see also Stevens, 1998). Third, forming only long-only GMVPs suggests that turnover (trading volume) and related transaction costs should be relatively small, since the range of the weights that each asset in the GMVP can have is bounded. Finally, the sufficient conditions that ensure obtaining long-only GMVPs allow for nonnegative covariances, which are very common in the US stock market (see, for example, Chan et al., 1999; and the 2011 Ibbotson SBBI valuation yearbook). It should be emphasized that some of the abovementioned advantages also apply to existing portfolio strategies in the literature, yet the combination of all of them together is why we expect our approach to perform well in practice.

We conduct an empirical analysis, to test the out-of-sample performance of a GMVP that is constructed using various block covariance matrices. In particular, we are interested in comparing the performance of the GMVP relative to that of the $1/N$ portfolio, which invests equally across N risky assets. We do this in light of DeMiguel et al. (2009a) and Duchin and Levy (2009), who show that, due to estimation errors, strategies based on mean-variance theory, however advanced they may be, do not consistently outperform the naïve $1/N$ portfolio.

Note that the $1/N$ portfolio can, in fact, be regarded as a GMVP that is constructed using a very restricted block covariance matrix, in which, whatever the number of blocks, all stocks have the same estimated variance (no matter what value this estimated variance takes), and the estimated between-block covariance and within-block covariances are all the same (no matter what value they take). Now, since the $1/N$ portfolio is such a restricted case, it is plausible to assume that if we construct GMVPs using less restricted covariance matrices, we will be able to obtain portfolio strategies that outperform the $1/N$ portfolio. We confirm the validity of this hypothesis in our empirical analysis.

We evaluate the performance of seven GMVPs that are constructed using block covariance matrices less restricted than that of the $1/N$ portfolio. In essence, as we move from one strategy to the next, we relax additional assumptions on the block covariance matrix. The seven strategies are listed in Table 1 and discussed in Subsection $3(i)$. We use four performance criteria for each portfolio strategy: the ex-post Sharpe ratio; the ex-post volatility (standard deviation); and two turnover (trading volume) measures that take into account turnover and transactions costs. We repeat the evaluation using 12 empirical datasets, which are listed in Table 2 and described

Table 1
List of Portfolio Strategies Considered

No.	*Abbreviation*	*Portfolio Strategy*
1	$1/N$	The $1/N$ portoflio that invests equally across N risky assets.
2	Diagonal	GMVP - the covariance matrix is diagonal. The variances are estimated using the sample variances. All covariances are set to zero.
3	Constant COV	GMVP - the covariance matrix includes the following elements: variances that are estimated using the sample variances; a constant covariance that is estimated using the sample covariances.
4	Constant COV SIM	GMVP - the covariance matrix includes the following elements: variances that are estimated using the sample variances; a constant covariance that is estimated using Sharpe's (1963) single-index model.
5	Block 1	GMVP - the covariance matrix is constructed using the block method. The variances are estimated using the sample variances. The within-block covariances are estimated using the sample covariances. The between-block covariance is set to zero.
6	Block SIM 1	GMVP - the covariance matrix is constructed using the block method. The variances are estimated using the sample variances. The within-block covariances are estimated using Sharpe's (1963) single-index model. The between-block covariance is set to zero.
7	Block 2	GMVP - the covariance matrix is constructed using the block method. The variances are estimated using the sample variances. The within-block covariances are estimated using the sample covariances. The between-block covariance is estimated using the sample covariances.
8	Block SIM 2	GMVP - the covariance matrix is constructed using the block method. The variances are estimated using the sample variances. The within-block covariances are estimated using Sharpe's (1963) single-index model. The between-block covariance is estimated using Sharpe's (1963) single-index model.
9	Value-weighted	The value-weighted portfolio implied by the market model.

Notes:
This table lists the various portfolio strategies we consider in the empirical analysis. The second column of the table gives the abbreviation used in referring to the strategy in Tables 3–6 and throughout the paper.

Table 2

List of Datasets Used

No.	*Abbreviation*	*Dataset and Source*	*Time Period*
		Stock Datasets	
1	Random 500	At the end of April of each year from 1968 to 2006, we randomly select 500 stocks listed on the NYSE, Amex and NASDAQ *Source: CRSP*	05/1963-04/2007
2	Random 250	At the end of April of each year from 1968 to 2006, we randomly select 250 stocks listed on the NYSE, Amex and NASDAQ *Source: CRSP*	05/1963-04/2007
3	International	At the end of April of each year from 1994 to 2006, we randomly select up to 35 listed stocks (35 when available) from each country included in the G-20 major economies (excluding the US and Canada) *Source: Compustat Global*	05/1989-04/2007
		Portfolio Datasets	
4	48 Industries	48 industry portfolios *Source: Kenneth French's website*	01/1963-12/2007
5	90	90 portfolios formed on size and book-to-market *Source: Kenneth French's website*	01/1963-12/2007
6	90 + 1	90 portfolios formed on size and book-to-market and the MKT portfolio *Source: Kenneth French's website*	01/1963-12/2007
7	90 + 3	90 portfolios formed on size and book-to-market and the MKT, SMB, and HML portfolios *Source: Kenneth French's website*	01/1963-12/2007
8	90 + 4	90 portfolios formed on size and book-to-market and the MKT, SMB, HML, and UMD portfolios *Source: Kenneth French's website*	01/1963-12/2007
9	20	20 portfolios formed on size and book-to-market *Source: Kenneth French's website*	01/1963-12/2007
10	20 + 1	20 portfolios formed on size and book-to-market and the MKT portfolio *Source: Kenneth French's website*	01/1963-12/2007
11	20 + 3	20 portfolios formed on size and book-to-market and the MKT, SMB, and HML portfolios *Source: Kenneth French's website*	01/1963-12/2007
12	20 + 4	20 portfolios formed on size and book-to-market and the MKT, SMB, HML, and UMD portfolios *Source: Kenneth French's website*	01/1963-12/2007

Notes:
This table lists the various datasets we use in the empirical analysis. The second column of the table gives the abbreviation used in referring to the dataset in Tables 3–6 and throughout the paper. Note that in the spirit of DeMiguel et al. (2009a), in datasets Nos. 5–8, we start with the 100 portfolios formed on size and book-to-market from Kenneth French's website and exclude the ten portfolios containing the smallest firms. Similarly, in datasets Nos. 9–12, we start with the 25 portfolios formed on size and book-to-market from Kenneth French's website and exclude the five portfolios containing the smallest firms.

in Sub-section 3(*ii*). Our results show that, in general, GMVPs that are constructed using covariance matrices less restricted than that of the $1/N$ portfolio, but still not complicated at all in terms of estimation, can outperform the $1/N$ portfolio. This is true across all datasets and performance criteria that we use.

DeMiguel et al. (2009a) conclude from their empirical study that the $1/N$ portfolio should serve as a benchmark to assess the performance of the various portfolio rules proposed in the literature. Our results indicate that a GMVP constructed using an easy to estimate block covariance matrix outperforms the $1/N$ portfolio, and can therefore better serve as a benchmark. Like those of Tu and Zhou (2011), our findings also suggest that mean-variance theory can still be useful in practice.

Lastly, even though one of the main features of our method is that it generates only long-only GMVPs, we also compare its performance to portfolio strategies proposed in the literature that allow for GMVPs with short sale positions. In these strategies, the GMVPs are either norm-constrained as in DeMiguel et al. (2009b), or constructed using covariance matrices that are based on: the constant correlation model of Elton and Gruber (1973) and Elton et al. (2003); factor models as in Chan et al. (1999); and the shrinkage method of Ledoit and Wolf (2003, 2004a and 2004b). Based on the ex-post Sharpe ratio and the turnover measures, our strategy performs at least as well as the strategies that allow for short sale positions. In 68% of the cases, our approach yields higher ex-post Sharpe ratios. In terms of turnover, it outperforms the strategies that allow for short sale positions in virtually all cases. Not surprisingly, due to the diversification effect of short sale positions, in 85% of the cases, the ex-post standard deviation of our method is higher than those of the strategies that allow for short positions. We believe that this set of results upholds our conclusion that the block structure for the covariance matrix of asset returns could have an impact on portfolio choice in practice.

The remainder of this paper proceeds as follows. In Section 2, we present the block structure and the related analytical results. Section 3 describes the portfolio strategies we evaluate in our empirical analysis, the datasets we use, and the methodology for evaluating the performance of the various portfolio strategies. The results of the empirical analysis appear in Section 4. Section 5 gives concluding remarks. In Appendix A, we generalize the block structure discussed in Section 2. In Appendix B, we give the proof for the main analytical result.

2. THE BLOCK STRUCTURE

For the block construct, one can divide the portfolio's stocks into any number of blocks. However, as shown in Table 3 and discussed in Sub-section 4(*i*), our empirical analysis indicates that the best performance results of the GMVP are generally obtained when the stocks are divided into only two blocks. Therefore, and for the sake of brevity, in this section, we present only the case of a two-block covariance matrix (the general block covariance matrix with more than two blocks is discussed in Appendix A). We assume a universe with n stocks. A covariance matrix $\mathbf{\Sigma}$ is said to be *two-block* if it has the following form:

$$\Sigma = \left(\begin{array}{cccc|cccc}
s_1^2 & \eta_1 & \cdots & \eta_1 & \eta & \cdots & & \eta \\
\eta_1 & s_2^2 & \ddots & \vdots & \vdots & \ddots & & \vdots \\
\vdots & \ddots & \ddots & & & & & \\
 & & & \eta_1 & & & & \\
\eta_1 & \cdots & \eta_1 & s_j^2 & \eta & \cdots & & \eta \\
\hline
\eta & \cdots & & \eta & s_{j+1}^2 & \eta_2 & \cdots & \eta_2 \\
\vdots & \ddots & & \vdots & \eta_2 & s_{j+2}^2 & \ddots & \vdots \\
 & & & & \vdots & \ddots & \ddots & \\
 & & & & & & & \eta_2 \\
\eta & \cdots & & \eta & \eta_2 & \cdots & \eta_2 & s_n^2
\end{array}\right)$$

Here j and n-j are the sizes of the two blocks (j and n-j are not necessarily equal), s_i^2 are the variances, η_1 and η_2 are the within-block covariances, and η is the between-block covariance.

Proposition 1 below characterizes sufficient conditions on η_1, η_2 and η under which the two-block matrix produces a long-only GMVP (in an unconstrained optimization). Note that the proposition's conditions guarantee obtaining not only a long-only GMVP but also a two-block matrix that is an invertible covariance matrix. Without further restrictions on η_1, η_2 and η, the conditions derived are a bit messy. However, restricting η_1, η_2 and η to be nonnegative generates simple and directly computable conditions: (i) the variance of each stock is greater than both its within-block covariance and its between-block covariance, and (ii) the within-block covariances are not smaller than the between-block covariance.

Proposition 1: Suppose that $\boldsymbol{\Sigma}$ is a two-block matrix. Then Σ produces a long-only GMVP, if the following conditions on η_1, η_2 and η hold:

$$\begin{aligned}
-|\eta_1^*| &< \eta_1 < \min\left(s_i^2\right), \quad i = 1, \ldots, j \\
-|\eta_2^*| &< \eta_2 < \min\left(s_i^2\right), \quad i = j+1, \ldots, n, \\
-|\eta_{12}^*| &< \eta \leq \min(\eta_1, \eta_2)
\end{aligned}$$

where $|\eta_1^*|$ and $|\eta_2^*|$ are respectively the unique solutions of the following equations:

$$|\eta_1^*| = 1 \Bigg/ \sum_{i=1}^{j} \frac{1}{s_i^2 + |\eta_1^*|} \quad \text{and} \quad |\eta_2^*| = 1 \Bigg/ \sum_{i=j+1}^{n} \frac{1}{s_i^2 + |\eta_2^*|},$$

and

$$|\eta_{12}^*| = +\sqrt{\left(\eta_1 + \frac{1}{\displaystyle\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\displaystyle\sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}\right)}.$$

Proposition 1 is proved in Appendix B.

When η_1, η_2 and η are restricted to be nonnegative, we get the following simple and directly computable sufficient conditions for which the GMVP is long-only:

Corollary: Suppose that $\boldsymbol{\Sigma}$ is a two-block matrix and the covariances η_1, η_2 and η are nonnegative. Then $\boldsymbol{\Sigma}$ produces a long-only GMVP, if the following conditions on η_1, η_2 and η hold:

$$\begin{aligned} &0 \leq \eta_1 < \min\left(s_i^2\right), \quad i = 1, \ldots, j \\ &0 \leq \eta_2 < \min\left(s_i^2\right), \quad i = j+1, \ldots, n \\ &0 \leq \eta \leq \min\left(\eta_1, \eta_2\right). \end{aligned}$$

As shown in the proof of Proposition 1, the expressions for the weight in the GMVP of a stock from the first and the second block are respectively:

$$\begin{aligned} w_i &= \frac{1}{s_i^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta)A_2}{A_1 + A_2 + (\eta_1 + \eta_2 - 2\eta)A_1 A_2}, \quad i = 1, \ldots, j \\ w_i &= \frac{1}{s_i^2 - \eta_2} \cdot \frac{1 + (\eta_1 - \eta)A_1}{A_1 + A_2 + (\eta_1 + \eta_2 - 2\eta)A_1 A_2}, \quad i = j+1, \ldots, n \end{aligned}$$

where $A_1 = \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}$, $A_2 = \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}$.

Assuming that the conditions for having a long-only GMVP hold, we can see that in each block the weights in the GMVP are biased towards the stocks with the relatively small variances. However, overall, the stocks with the smallest variances are not necessarily also those with the largest weights in the GMVP. That is because of two more effects that need to be taken into consideration. The first is the differences between the stock variances and the within-block covariances. The second is the differences between the within-block covariances and the between-block covariances.[7] We can also see that the conditions of Proposition 1 are flexible enough not to generate 'corner' solutions with zero weights in many assets.

3. EMPIRICAL ANALYSIS

The main purpose of our empirical analysis is to compare the performance of a GMVP that is constructed using a block covariance matrix to that of the $1/N$ portfolio, which was recently suggested as a benchmark to assess the performance of various portfolio rules proposed in the literature. DeMiguel et al. (2009a) present a comprehensive study in which they evaluate the performance of 14 models of portfolio selection across seven different datasets. Their main finding is that none of the 14 evaluated models, not even the more advanced ones, consistently outperforms the $1/N$ portfolio.[8]

7 Assume that we have four stocks, two in each block. In the first block, $s_1^2 = 0.35, s_2^2 = 0.37, \eta_1 = 0.27$; in the second block, $s_3^2 = 0.4, s_4^2 = 0.5, \eta_2 = 0.01$; and the between-block covariance is $\eta = 0$. The GMVP weights here are: $w_1 = 0.23, w_2 = 0.19$, $w_3 = 0.32, w_4 = 0.26$ — a larger fraction of the GMVP is invested in the stocks with the larger variances.

8 The advanced techniques evaluated in DeMiguel et al. (2009a) include Bayesian models for estimation, the MacKinlay and Pastor (2000) missing factor model, the Kan and Zhou (2007) three-fund model, and the multi-prior model of Garlappi et al. (2007). In our study, we do not consider any of these strategies, or the other portfolio strategies evaluated by DeMiguel et al. (2009a), since they show none of them consistently outperforms the $1/N$ portfolio.

Duchin and Levy (2009) show that that the 1/N portfolio outperforms relatively small mean-variance theory-based portfolios.

We notice that the 1/N portfolio can, in fact, be regarded as a GMVP that is constructed using a very restricted block covariance matrix. One in which, whatever the number of blocks, all stocks have the same estimated variance (no matter what value this estimated variance takes), and the estimated between-block covariance and within-block covariances are all the same (no matter what value they take). Therefore, our goal is to test whether once we use less restricted block covariance matrices, we are able to construct a GMVP that can perform better than the 1/N portfolio. Since we would like the new portfolio strategy to replace the 1/N portfolio as a benchmark to assess the performance of various portfolio rules, we make sure to look at block-covariance matrices that are less restricted but still not at all complicated in terms of estimation. They are described in the next sub-section.

(i) Description of the Portfolio Strategies Considered

We include the following portfolio strategies in the empirical analysis (the abbreviations are as in Table 1).[9]

1/N: This portfolio invests equally across N stocks. As mentioned above, it can be regarded as a GMVP that is constructed using a very restricted block covariance matrix.

Diagonal: Here we have a GMVP that is constructed using a diagonal covariance matrix. The variances are estimated using the traditional sample variances, and all covariances are set to zero. It is easy to see that the diagonal covariance matrix is another special case of a block covariance matrix in which we relax the 1/N assumption that all stocks have the same estimated variance and tighten the assumption that all covariances take the same value, by setting this value to zero.

Constant COV: Here we have a GMVP that is constructed using the following covariance matrix. The variances are estimated using the traditional sample variances. All covariances are estimated using the traditional sample covariances, and the average estimated covariance is then calculated. To meet the sufficient conditions for obtaining a long-only GMVP, if the average estimated covariance is smaller than the smallest estimated variance, all covariances are set to the value of the average estimated covariance. If the average estimated covariance is not smaller than the smallest estimated variance, all covariances are set to 50% of the smallest estimated variance.[10] Yet again it is easy to see that the covariance matrix here is a special case of the block covariance matrix. Contrary to the diagonal covariance matrix, here we do not set

9 We will be using 'stocks' in describing the portfolio strategies considered. Note that the portfolio strategies are tested across stock datasets as well as stock portfolio datasets. Hence, when the stock portfolio datasets are in use, 'stocks' in fact stand for 'stock portfolios.'

10 We tried a range of values from 0% to 90% (in increments of 10%). In all 12 datasets, except the international dataset, the value of the coefficient had virtually no effect on the performance results of the portfolio strategy (so we arbitrarily chose a coefficient of 50%). The reason is that in most cases we met the sufficient conditions for obtaining a long-only GMVP without the need to use this coefficient at all. The same also applies to the instances we needed similar coefficients for the next portfolio strategies: 'Constant COV SIM', 'Block 1', 'Block SIM 1', 'Block 2' and 'Block SIM 2'.

the constant covariance to zero. Instead, we estimate it using the traditional sample covariances.

Constant COV SIM: Here the GMVP is constructed using a covariance matrix very similar to that of the 'Constant COV.' The only difference is that here, instead of calculating the average estimated covariance based on the traditional sample covariances, we estimate the covariance between any pair of stocks using Sharpe's (1963) single-index model, and then calculate the average of these estimated covariances. Mostly, the CRSP (Center for Research in Security Prices) equally weighted portfolio is used as the 'market' portfolio for the single-index model.[11]

Block 1: Here the GMVP is constructed using the following block covariance matrix. The stocks are ranked based on their estimated traditional sample variances (from the largest to the smallest), and then divided into the blocks. For instance, in the two-block matrix case, the half of the stocks with the larger estimated variances forms the first block and the half of the stocks with the smaller estimated variances forms the second block. To test how the number of blocks affects the portfolio's performance, we divide the stocks into up to eight blocks. As to estimating the elements of the covariance matrix, the variances are estimated using the traditional sample variances. The within-block covariances are estimated using the traditional sample covariances. That is, in each block we estimate the covariances between the stocks in that block using the traditional sample covariances, and then calculate the average estimated covariance for the block. To meet the sufficient conditions for obtaining a long-only GMVP, if the average estimated covariance of the block is smaller than the smallest estimated variance in that block, the within-block covariance of that block is set to the value of the average estimated covariance of that block. If the average estimated covariance of the block is not smaller than the smallest estimated variance in that block, the within-block covariance of that block is set to 50% of the smallest estimated variance in that block. Finally, the between-block covariance is set to zero. Clearly, this block covariance matrix is less restricted than that of the $1/N$ portfolio. We relax the assumptions that all stocks have the same variance and that all the covariances take the same value.

Block SIM 1: Here the GMVP is constructed using a block covariance matrix very similar to that of the 'Block 1' strategy. The only difference is that here, instead of calculating the average estimated covariance of each block based on the traditional sample covariances, we estimate the covariance between any pair of stocks in that block using Sharpe's (1963) single-index model, and then calculate the average of these estimated covariances. Mostly, the CRSP equally weighted portfolio is used as the 'market' portfolio for the single index model.

Block 2: Here the GMVP is constructed using a block covariance matrix very similar to that of the 'Block 1' strategy. The only difference is that here, instead of setting the between-block covariance to zero, we estimate it using the traditional sample covariances. That is, for each pair of stocks from different blocks we estimate the covariance using the traditional sample covariance, and then calculate the average

11 The only exception is when we use the international dataset. Then we use the MSCI world index (excluding the US), as the 'market' portfolio. The same is also true for the portfolio strategies 'Block SIM 1' and 'Block SIM 2', which are discussed later.

of these estimated covariances. To meet the sufficient conditions for obtaining a long-only GMVP, if the average estimated covariance between pairs of stocks from different blocks is not greater than the smallest within-block covariance, the between-block covariance is set to the value of the calculated average estimated covariance. If the average estimated covariance between pairs of stocks from different blocks is greater than the smallest within-block covariance, the between-block covariance is set to 50% of the smallest within-block covariance. Clearly, this block covariance matrix is even less restricted, as we do not impose a zero between-block covariance.

Block SIM 2: Here the GMVP is constructed using a block covariance matrix very similar to that of the 'Block SIM 1' strategy. The only difference is that here, instead of setting the between-block covariance to zero, we estimate it using Sharpe's (1963) single-index model. That is, for each pair of stocks from different blocks we estimate the covariance using Sharpe's (1963) single-index model, and then calculate the average of these estimated covariances. To meet the sufficient conditions for obtaining a long-only GMVP, if the average estimated covariance between pairs of stocks from different blocks is not greater than the smallest within-block covariance, the between-block covariance is set to the value of the calculated average estimated covariance. If the average estimated covariance between pairs of stocks from different blocks is greater than the smallest within-block covariance, the between-block covariance is set to 50% of the smallest within-block covariance. Clearly, this block covariance matrix is even less restricted than that of 'Block SIM 1', as we do not impose a zero between-block covariance.

Value-weighted: The optimal strategy in a CAPM world is the value-weighted market portfolio. So, for each of the datasets we identify a benchmark 'market' portfolio. For the stock portfolio datasets, we use the CRSP value-weighted portfolio, and for the stock datasets, we calculate the respective value-weighted portfolios.

Next, we describe the various datasets across which we evaluate the performance of the aforementioned portfolio strategies.

(ii) Description of the Empirical Datasets

We consider both stock datasets and stock portfolio datasets. The main difference between these two types of datasets is that in portfolios of stocks the variances are smoother, since generally portfolio average returns have smaller variances than individual stocks. To make sure that our results are not affected by this difference, we use both types of datasets. The following datasets are used in our empirical analysis (we use the same abbreviations as in Table 2):

(a) Stock Datasets

Random 500: Similarly to Chan et al. (1999) and Jagannathan and Ma (2003), at the end of April of each year from 1968 to 2006, we randomly select 500 stocks listed on the NYSE, Amex and NASDAQ (we consider only stocks above the second decile of market capitalization based on NYSE breakpoints, and with prices greater than $5). The monthly stock returns are extracted from the CRSP database. The data span from

May 1963 to April 2007 (including the first estimation periods and the last out-of-sample period).

Random 250: This dataset is very similar to 'Random 500'. The only difference is that here each year we randomly select 250 stocks instead of 500 stocks.

International: At the end of April of each year from 1994 to 2006, we randomly select up to 35 listed stocks (35 when available) from each country included in the G-20 major economies, excluding the US and Canada (out of the entire pool of stocks, we consider only those above the second decile of market capitalization). The stock returns are extracted from the Compustat Global database. The data span from May 1989 to April 2007 (including the first estimation periods and the last out-of-sample period).

(b) Stock Portfolio Datasets

48 Industries: The data consist of monthly returns on 48 industry portfolios. The portfolio returns are extracted from Kenneth French's website and span from January 1963 to December 2007 (including the first estimation periods and the last out-of-sample period).

90, 90 + 1, 90 + 3, and 90 + 4: The dataset '90' consists of monthly returns on 90 portfolios formed on size and book-to- market (we start with the 100 portfolios formed on size and book-to-market from Kenneth French's website, and in the spirit of DeMiguel et al., 2009a, exclude the ten portfolios containing the smallest firms). In the dataset '90 + 1', we augment the '90' dataset with the US market portfolio (MKT). In the '90 + 3' dataset, we augment the '90 + 1' dataset with HML, a zero-cost portfolio that is long in high book-to-market stocks and short in low book-to-market stocks, and SMB, a zero-cost portfolio that is long in small-cap stocks and short in large-cap stocks. In the '90 + 4' dataset, we augment the '90 + 3' dataset with the zero-cost momentum portfolio UMD. In all cases, the monthly portfolio returns are extracted from Kenneth French's website and span from January 1963 to December 2007 (including the first estimation periods and the last out-of-sample period).

20, 20 + 1, 20 + 3, and 20 + 4: This group of four datasets is very similar to the '90', '90 + 1', '90 + 3', and '90 + 4' group. The only difference is that here we start with the 25 portfolios formed on size and book-to-market from Kenneth French's website and, in the spirit of DeMiguel et al. (2009a), exclude the five portfolios containing the smallest firms.

In the next sub-section, we present the methodology and the performance criteria we employ to study the performance of each of the portfolio strategies from Sub-section 3(i) across the aforementioned datasets.

(iii) Methodology and Performance Criteria

Following Chan et al. (1999), Bengtsson and Holst (2002), Jagannathan and Ma (2003), Ledoit and Wolf (2003) and DeMiguel et al. (2009a), our empirical analysis relies on the 'rolling-sample' approach. Thus, in a nutshell, given a T-month long dataset of monthly stock returns, we choose an estimation window (in-sample period) of length $L = 60$ months.[12] In each month t, starting from $t = L + 1$, we use the returns in the previous L months to estimate the parameters needed for each of our covariance matrix estimators. Each of the covariance matrix estimators are then used to determine the weights in the corresponding GMVPs. These weights are then used to compute the monthly returns on each of the GMVPs in the out-of-sample period from $t = L + 1$ till $t = L + k = L + 12$.[13] In month $t = L + k + 1$, we start the whole process all over again; that is, we use the returns in the previous L months to estimate the various covariance matrix estimators, determine the weights in the corresponding GMVPs, and compute the monthly returns on the GMVPs in the out-of-sample period of $k = 12$ months. The process is continued by adding the returns of the next k months in the dataset and dropping the earliest k returns, until the end of the dataset is reached.[14] The outcome of this procedure is a series of T-L monthly out-of-sample (ex-post) returns generated by each of our GMVPs. In the same manner, but of course without the need to estimate the covariance matrix, we derive the series of the ex-post returns on the $1/N$ portfolio and the value-weighted portfolio. Lastly, we subtract from each of the ex-post return series the corresponding T-L one-month T-bill returns (extracted from Kenneth French's website). In this way we obtain a series of ex-post excess returns generated by each of the portfolio strategies listed in Table 1, for each of the empirical datasets listed in Table 2. These series are used to compute the performance criteria that we employ in the analysis: the ex-post Sharpe ratio, the portfolio turnover, the return-loss of each strategy, and the ex-post volatility (standard deviation).

We measure the ex-post Sharpe ratio of strategy i, SR_i, which is defined as $\mathrm{SR}_i = \mathrm{Z}_i/\sigma_i$, where Z_i and σ_i denote respectively the ex-post expected excess return and the ex-post standard deviation of series i. To test whether the Sharpe ratios of two portfolio strategies are statistically different, we use the studentized time series bootstrap method introduced in Ledoit and Wolf (2008).

To get a sense of the amount of trading required to implement each portfolio strategy, we compute the portfolio turnover and the return-loss, net of transactions costs, with respect to the $1/N$ portfolio, as in DeMiguel et al. (2009a).

The portfolio turnover is defined as the average sum of the absolute value of the trades across the N available assets:

$$\text{Turnover} = \frac{1}{T-L}\sum_{t=1}^{T-L}\sum_{j=1}^{N}\left(|\hat{w}_{i,j,t+1} - \hat{w}_{i,j,t^+}|\right),$$

where $\hat{w}_{i,j,t}$ denotes the portfolio weight in asset j at time t under portfolio strategy i; $\hat{w}_{i,j,t^+}$ denotes the portfolio weight due to changes in asset prices between t and $t+1$; and $\hat{w}_{i,j,t+1}$ denotes the desired portfolio weight at time $t+1$. As explained in

12 Chan et al. (1999) and Jagannathan and Ma (2003) also use an in-sample period of 60 months.

13 Chan et al. (1999), Jagannathan and Ma (2003) and Ledoit and Wolf (2003) also use an out-of-sample period of 12 months.

14 As an aside, when the stock datasets are used, each time we construct the portfolios, we use only stocks whose returns cover the entire respective in-sample period. When a monthly return is missing in the out-of-sample period, the CRSP equally weighted market return of that month replaces it. We are aware of the fact that this widely-followed procedure (see, for instance, Jagannathan and Ma, 2003) introduces survivorship bias into the estimation procedure. However, since the survivorship bias is common to all the compared estimators, we do not consider this a significant problem. With the stock portfolio datasets, there is no problem of missing data.

DeMiguel et al. (2009a), for example, in the case of the $1/N$ portfolio, $\hat{w}_{i,j,t} = \hat{w}_{i,j,t+1} = 1/N$, but $\hat{w}_{i,j,t^+}$ may be different due to changes in asset prices between t and $t+1$.

To calculate for each strategy the return-loss, net of transactions costs, with respect to the $1/N$ portfolio, we follow the steps in DeMiguel et al. (2009a). Let $R_{i,p}$ be the return from strategy i on the portfolio of N assets due to changes in asset prices between t and $t+1$; that is, $R_{i,p} = \sum_{J=1}^{N} R_{j,t+1}\hat{w}_{i,j,t}$. Denote by c the proportional transaction cost. Therefore, we can write the evolution of wealth for strategy i as follows:

$$W_{i,t+1} = W_{i,t}(1 + R_{i,p})\left(1 - c \times \sum_{j=1}^{N} |\hat{w}_{i,j,t+1} - \hat{w}_{i,j,t^+}|\right),$$

with the return net of transactions costs given by $\frac{W_{i,t+1}}{W_{i,t}} - 1$. In our empirical analysis, like DeMiguel et al. (2009a), we set the proportional transactions costs, c, equal to 50 basis points per transaction. The return-loss, net of the transactions costs, with respect to the $1/N$ portfolio is defined as the additional return needed for strategy i to perform as well as the $1/N$ portfolio in terms of the ex-post Sharpe ratio. To compute the return-loss, assume μ_N and σ_N are the monthly ex-post mean and standard deviation of the net returns from the $1/N$ portfolio, and μ_i and σ_i are the corresponding quantities for strategy i. Then, the return-loss from strategy i is:

$$\text{return} - \text{loss}_i = \frac{\mu_N}{\sigma_N} \times \sigma_i - \mu_i.$$

As we evaluate the performance of portfolio strategies that are based on GMVPs, we also measure the ex-post volatility of strategy i, which is defined as the ex-post standard deviation of series i. To test whether the standard deviations of two portfolio strategies are statistically different, we use an F-test.[15]

4. RESULTS OF THE EMPIRICAL ANALYSIS

In this section, we present the results of our empirical analysis. First, we study the effect of the number of blocks in the covariance matrix on the performance of the GMVP (Table 3). Then, we compare the performance of the portfolio strategies listed in Table 1 to that of the $1/N$ portfolio. For each strategy, we compute across all the datasets listed in Table 2, the ex-post Sharpe ratio (Table 4), the turnover and the return-loss measure (Table 5), and the ex-post standard deviation (Table 6). The section ends with a description of the empirical comparison we make between the performance of our approach, which is set to generate only long-only GMVPs, and strategies proposed in the literature that allow for GMVPs with short sale positions (Tables 7–10).

(i) Number of Blocks and the GMVP's Performance

Table 3 describes the effect of the number of blocks in the block covariance matrix on the ex-post Sharpe ratio of the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2', and 'Block SIM 2' (the other portfolio strategies listed in Table 1, by construction,

15 Ledoit and Wolf (2011) provide an alternative approach to test for significant differences in portfolio volatility.

Table 3

Sharpe Ratio as a Function of Number of Blocks

	Stock Datasets			*Portfolio Datasets*								
	Random 500	*Random 250*	*International*	*48 Industries*	*90*	*90 + 1*	*90 + 3*	*90 + 4*	*20*	*20 + 1*	*20 + 3*	*20 + 4*
Block 1												
2 blocks	0.59	0.61	0.73	0.49	0.49	0.48	0.44	0.45	0.48	0.45	0.33	0.39
3 blocks	0.58	0.58	0.72	0.50	0.48	0.47	0.43	0.44	0.48	0.44	0.31	0.37
4 blocks	0.58	0.58	0.75	0.52	0.47	0.46	0.41	0.43	0.48	0.44	0.31	0.37
5 blocks	0.58	0.58	0.77	0.51	0.46	0.45	0.40	0.43				
6 blocks	0.58	0.59	0.76	0.51	0.46	0.45	0.41	0.43				
7 blocks	0.58	0.59	0.78		0.47	0.46	0.41	0.44				
8 blocks	0.58	0.58	0.76		0.47	0.46	0.40	0.43				
Block SIM 1												
2 blocks	0.62	0.60	0.72	0.49	0.50	0.49	0.45	0.47	0.48	0.46	0.34	0.38
3 blocks	0.58	0.61	0.75	0.50	0.47	0.47	0.44	0.46	0.48	0.45	0.26	0.35
4 blocks	0.59	0.61	0.73	0.51	0.48	0.48	0.43	0.44	0.48	0.44	0.31	0.33
5 blocks	0.59	0.62	0.72	0.52	0.49	0.48	0.41	0.44				
6 blocks	0.60	0.62	0.71	0.50	0.49	0.48	0.42	0.44				
7 blocks	0.59	0.61	0.71		0.49	0.48	0.42	0.43				
8 blocks	0.60	0.61	0.71		0.48	0.47	0.42	0.43				
Block 2												
2 blocks	0.61	0.63	0.77	0.50	0.49	0.48	0.44	0.45	0.48	0.45	0.31	0.38
3 blocks	0.59	0.62	0.76	0.52	0.49	0.47	0.43	0.44	0.48	0.44	0.27	0.33
4 blocks	0.58	0.63	0.79	0.54	0.48	0.47	0.41	0.43	0.48	0.44	0.25	0.33
5 blocks	0.60	0.63	0.82	0.53	0.47	0.45	0.40	0.43				
6 blocks	0.59	0.64	0.77	0.53	0.47	0.45	0.40	0.43				
7 blocks	0.60	0.64	0.81		0.48	0.46	0.40	0.43				
8 blocks	0.60	0.64	0.79		0.48	0.46	0.39	0.41				

Block SIM 2												
2 blocks	0.63	0.63	0.76	0.50	0.51	0.50	0.45	0.47	0.49	0.46	0.32	0.36
3 blocks	0.60	0.64	0.77	0.51	0.48	0.47	0.44	0.46	0.48	0.45	0.21	0.32
4 blocks	0.58	0.63	0.76	0.52	0.49	0.48	0.42	0.43	0.48	0.44	0.27	0.29
5 blocks	0.61	0.65	0.76	0.54	0.49	0.48	0.40	0.43				
6 blocks	0.60	0.64	0.74	0.52	0.49	0.48	0.41	0.43				
7 blocks	0.59	0.63	0.74		0.49	0.48	0.40	0.40				
8 blocks	0.61	0.63	0.77		0.49	0.48	0.40	0.39				

Notes:
For each of the datasets listed in Table 2, this table reports the ex-post Sharpe ratios of the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2', and 'Block SIM 2', when a two-block covariance matrix is used to construct the GMVP, a three-block covariance matrix is used to construct the GMVP and so on up to an eight-block covariance matrix. When the 48 Industries dataset is used, we reach only a six-block covariance matrix, and when the '20', '20 + 1', '20 + 3', and '20 + 4' datasets are used, we reach only a four-block covariance matrix, because then we construct the GMVPs using a smaller number of assets. The Sharpe ratios are annualized through multiplication by $\sqrt{12}$.

are not affected by the number of blocks). When we use the stock datasets and the portfolio datasets '90', '90 + 1', '90 + 3', and '90 + 4,' we evaluate each of the four aforementioned portfolio strategies using a two-block covariance matrix, a three-block covariance matrix and so on up to an eight-block covariance matrix. When the '48 Industries' dataset is used, we reach only a six-block covariance matrix, and when the '20', '20 + 1', '20 + 3', and '20 + 4' datasets are used, we reach only a four-block covariance matrix, because then we construct the GMVPs using a smaller number of assets.

We can see in Table 3 that, in general, the four portfolio strategies yield the highest ex-post Sharpe ratios when a two-block covariance matrix is used. Moreover, even in those cases in which dividing the assets into more than two blocks yields higher Sharpe-ratios the differences compared to the two-block case are mostly only marginal. Overall, it seems that the number of blocks has no real effect on the ex-post Sharpe ratios of the portfolio strategies. We also test whether the number of blocks in the covariance matrix affects the ex-post standard deviations of the four aforementioned portfolio strategies. As the insights from this set of tests are very similar to those of the Sharpe ratio tests, to save space, they are not reported in the paper, but are available upon request from the authors.

Consistent with the above discussion, Tables 4–6 report the performance measures of the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2', and 'Block SIM 2' only for the case in which the two-block covariance matrix is used to construct the corresponding GMVPs.

(ii) Sharpe Ratios

Table 4 gives the ex-post Sharpe ratio for each portfolio strategy evaluated in our empirical analysis. We can see that in eight out of the 12 datasets used, all the portfolio strategies in which the GMVP is constructed using a block covariance matrix less restricted than that of the $1/N$ portfolio achieve higher Sharpe ratios than the $1/N$ portfolio. Only in two datasets the $1/N$ portfolio yields the highest Sharpe ratio, though with p-values of the difference between the Sharpe ratio of the $1/N$ portfolio and each of the other portfolio strategies that are always greater than 11.7%. In the remaining two datasets, all portfolio strategies perform virtually the same, apart from the value-weighted portfolio with its notably poor performance across all datasets.

Specifically, we are interested in the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2', and 'Block SIM 2'. That is because in these strategies the GMVP is constructed using a significantly less restricted block covariance matrix than that of the $1/N$ portfolio, yet still an easy one to calculate. Overall, we can say that these strategies consistently outperform the $1/N$ portfolio in terms of the ex-post Sharpe ratio. The p-value of the difference between the Sharpe ratio of each of these strategies and that of the $1/N$ portfolio is in many cases smaller than 5%.[16]

The discussion so far did not consider transactions costs. In the next sub-section, we show that our new portfolio approach also outperforms the $1/N$ portfolio in the presence of transactions costs.

16 In the international dataset, the p-values are relatively large, since we have fewer observations compared to the other empirical datasets.

Table 4

Sharpe Ratios for Empirical Data

	Stock Datasets			*Portfolio Datasets*								
Portfolio Strategy	*Random 500*	*Random 250*	*International*	*48 Industries*	*90*	*90+ 1*	*90+ 3*	*90+ 4*	*20*	*20+ 1*	*20+ 3*	*20+ 4*
1/*N*	0.50	0.47	0.64	0.42	0.44	0.44	0.44	0.44	0.45	0.43	0.41	0.43
Diagonal	0.58	0.58	0.78	0.46	0.48	0.47	0.45	0.46	0.48	0.45	0.38	0.42
	(0.071)	(0.013)	(0.226)	(0.032)	(0.025)	(0.039)	(0.364)	(0.179)	(0.050)	(0.172)	(0.299)	(0.728)
Constant COV	0.58	0.59	0.83	0.48	0.48	0.47	0.44	0.45	0.49	0.45	0.35	0.38
	(0.102)	(0.021)	(0.109)	(0.028)	(0.022)	(0.046)	(0.785)	(0.470)	(0.048)	(0.196)	(0.190)	(0.343)
Constant COV SIM	0.58	0.60	0.81	0.48	0.48	0.47	0.44	0.44	0.49	0.45	0.34	0.39
	(0.084)	(0.013)	(0.135)	(0.031)	(0.037)	(0.094)	(0.842)	(0.783)	(0.049)	(0.327)	(0.117)	(0.620)
Block 1	0.59	0.61	0.73	0.49	0.49	0.48	0.44	0.45	0.48	0.45	0.33	0.39
	(0.105)	(0.002)	(0.536)	(0.022)	(0.015)	(0.038)	(0.982)	(0.619)	(0.025)	(0.197)	(0.160)	(0.458)
Block SIM 1	0.62	0.60	0.72	0.49	0.50	0.49	0.45	0.47	0.48	0.46	0.34	0.38
	(0.071)	(0.002)	(0.631)	(0.017)	(0.043)	(0.031)	(0.706)	(0.417)	(0.099)	(0.143)	(0.176)	(0.389)
Block 2	0.61	0.63	0.77	0.50	0.49	0.48	0.44	0.45	0.48	0.45	0.31	0.38
	(0.107)	(0.001)	(0.417)	(0.018)	(0.023)	(0.055)	(0.994)	(0.630)	(0.042)	(0.241)	(0.138)	(0.424)
Block SIM 2	0.63	0.63	0.76	0.50	0.51	0.50	0.45	0.47	0.49	0.46	0.32	0.36
	(0.075)	(0.001)	(0.490)	(0.013)	(0.027)	(0.038)	(0.756)	(0.444)	(0.121)	(0.146)	(0.149)	(0.340)
Value-weighted	0.38	0.41	0.23	0.34	0.34	0.34	0.34	0.34	0.34	0.34	0.34	0.34
	(0.059)	(0.468)	(0.129)	(0.439)	(0.124)	(0.138)	(0.176)	(0.161)	(0.099)	(0.171)	(0.308)	(0.210)

Notes:
For each of the datasets listed in Table 2, this table reports the ex-post Sharpe ratios of the portfolio strategies listed in Table 1. In the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2', and 'Block SIM 2', a two-block covariance matrix is used to construct the GMVP. In parentheses is the p-value of the difference between the Sharpe ratio of each strategy and that of the $1/N$ portfolio, which is computed using the studentized time series bootstrap method introduced in Ledoit and Wolf (2008). The Sharpe ratios are annualized through multiplication by $\sqrt{12}$.

Table 5

Portfolio Turnovers for Empirical Data

	Stock Datasets			*Portfolio Datasets*								
Portfolio Strategy	*Random 500*	*Random 250*	*International*	*48 Industries*	*90*	*90+ 1*	*90+ 3*	*90+ 4*	*20*	*20+ 1*	*20+ 3*	*20+ 4*
Panel A: Turnover of each Portfolio Strategy												
$1/N$	0.066	0.066	0.085	0.027	0.023	0.022	0.023	0.023	0.016	0.016	0.018	0.019
Diagonal	0.054	0.054	0.065	0.040	0.032	0.032	0.033	0.034	0.022	0.022	0.029	0.031
Constant COV	0.063	0.063	0.076	0.035	0.032	0.032	0.032	0.032	0.023	0.023	0.033	0.030
Constant COV SIM	0.063	0.063	0.077	0.035	0.035	0.035	0.036	0.035	0.026	0.027	0.028	0.035
Block 1	0.050	0.050	0.060	0.048	0.049	0.048	0.042	0.042	0.030	0.031	0.042	0.041
Block SIM 1	0.049	0.049	0.064	0.049	0.052	0.050	0.043	0.042	0.047	0.042	0.037	0.041
Block 2	0.048	0.047	0.057	0.047	0.049	0.048	0.041	0.042	0.031	0.032	0.042	0.040
Block SIM 2	0.047	0.046	0.059	0.047	0.051	0.049	0.041	0.040	0.048	0.044	0.036	0.040
Value-weighted	0.052	0.052	0.079	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Panel B: Return-Loss Relative to the 1/N Portfolio												
Diagonal	−0.022	−0.027	−0.028	−0.012	−0.008	−0.007	−0.007	−0.009	−0.007	−0.005	−0.006	−0.010
Constant COV	−0.008	−0.007	−0.023	0.001	0.005	0.005	0.000	0.000	0.002	0.000	0.007	0.008
Constant COV SIM	−0.008	−0.007	−0.022	0.001	0.007	0.007	0.001	0.001	0.003	0.000	0.004	0.004
Block 1	−0.026	−0.034	−0.022	−0.019	−0.009	−0.008	−0.010	−0.013	−0.007	−0.005	−0.006	−0.012
Block SIM 1	−0.031	−0.033	−0.017	−0.019	−0.012	−0.011	−0.012	−0.015	−0.006	−0.006	−0.008	−0.012
Block 2	−0.030	−0.038	−0.029	−0.022	−0.011	−0.009	−0.011	−0.014	−0.008	−0.006	−0.006	−0.014
Block SIM 2	−0.033	−0.038	−0.024	−0.022	−0.013	−0.012	−0.013	−0.016	−0.007	−0.006	−0.008	−0.012
Value-weighted	0.014	0.006	0.169	−0.002	0.006	0.006	0.006	0.007	0.008	0.006	0.008	0.013

Notes:
For each of the datasets listed in Table 2, Panel A reports the turnovers of the portfolio strategies listed in Table 1, and Panel B reports the return-loss, which is the extra return a strategy needs to provide for its Sharpe ratio to equal that of the 1/*N* portfolio in the presence of proportional transactions costs of 50 basis points. Note that a negative return-loss is obtained when the Sharpe ratio of a portfolio strategy is higher than that of the 1/*N* portfolio in the presence of the proportional transactions costs. In the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2', and 'Block SIM 2', a two-block covariance matrix is used to construct the GMVP. The return-losses are annualized through multiplication by 12.

(iii) Turnover and Return-Loss

Table 5 reports the results for portfolio turnover, our second performance criterion. In Panel A of Table 5, we have the raw turnover for each portfolio strategy. We can see that for the stock datasets, the turnover of the $1/N$ portfolio is higher than that of all the portfolio strategies in which the GMVP is constructed using a less restricted block covariance matrix than that of the $1/N$ portfolio, whereas for the stock portfolio datasets the opposite is true. But, more importantly, in essence, all the raw turnovers are of the same order of magnitude, especially compared to the turnover results of DeMiguel et al. (2009a), who report turnovers for their evaluated portfolio strategies exceeding that of the $1/N$ portfolio by several orders of magnitude.[17] This may lead to the conclusion that the strong performance of our new portfolio approach compared to the $1/N$ portfolio in terms of Sharpe ratio is not at the expense of turnover.

The results reported in Panel B of Table 5 confirm the last conclusion. If we focus on the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2', and 'Block SIM 2', we see that they have a negative return-loss in all datasets. This implies that in all cases, these portfolio strategies attain a higher Sharpe ratio than the $1/N$ portfolio after controlling for transactions costs. Note that in many cases the return-loss of the $1/N$ portfolio compared to our portfolio approach is non-trivial and can reach 2 to 3.8% in annual terms. As an aside, in all datasets, the GMVP constructed using the diagonal covariance matrix also beats the $1/N$ portfolio, in the presence of transactions costs, and the value-weighted portfolio continues to perform poorly.

(iv) Standard Deviations

So far we have seen that, in terms of the ex-post Sharpe ratio, our new portfolio approach, in general, outperforms the $1/N$ portfolio. This is true both when transactions costs are not considered, as well as when they are taken into account. Table 6 contains the results for portfolio volatility, our third measure of performance.

The comparison of the ex-post standard deviations in Table 6 confirms the conclusions from the analysis of Sharpe ratios. In all cases, the volatility of the GMVPs that are constructed using a covariance matrix less restricted than that of the $1/N$ portfolio is lower than the volatility of the $1/N$ portfolio. In most cases, the p-value of the difference between the standard deviation of each strategy and that of the $1/N$ portfolio is smaller than 0.1%. It can be noticed that in terms of volatility the value-weighted portfolio performs better than the $1/N$ portfolio.

From the above discussion, which relies on several performance criteria, we conclude that our new approach to portfolio choice consistently outperforms the $1/N$ portfolio. We believe that the explanation for our findings is the following. We notice that the $1/N$ portfolio can be viewed as a GMVP that is constructed using a very restricted block covariance matrix. Therefore, once we use less restricted block covariance matrices, we are able to construct a GMVP that can perform better than the $1/N$ portfolio.

17 The raw turnover measure of the value-weighted portfolio is by construction zero when we use the stock portfolio datasets, since then our benchmark 'market' portfolio is the CRSP value-weighted portfolio.

Table 6

Volatility for Empirical Data

Portfolio	*Stock Datasets*			*Portfolio Datasets*								
Strategy	*Random 500*	*Random 250*	*International*	*48 Industries*	*90*	*90 + 1*	*90 + 3*	*90 + 4*	*20*	*20 + 1*	*20 + 3*	*20 + 4*
$1/N$	0.177	0.178	0.176	0.195	0.179	0.179	0.175	0.173	0.176	0.174	0.160	0.153
Diagonal	0.145	0.144	0.132	0.180	0.171	0.171	0.158	0.154	0.169	0.168	0.131	0.121
	(<0.001)	(<0.001)	(<0.001)	(0.077)	(0.315)	(0.309)	(0.024)	(0.010)	(0.395)	(0.383)	(<0.001)	(<0.001)
Constant COV	0.141	0.139	0.128	0.174	0.169	0.169	0.151	0.146	0.167	0.165	0.119	0.108
	(<0.001)	(<0.001)	(<0.001)	(0.012)	(0.192)	(0.186)	(<0.001)	(<0.001)	(0.250)	(0.240)	(<0.001)	(<0.001)
Constant COV SIM	0.141	0.139	0.128	0.174	0.169	0.168	0.150	0.145	0.166	0.165	0.115	0.106
	(<0.001)	(<0.001)	(<0.001)	(0.012)	(0.171)	(0.164)	(<0.001)	(<0.001)	(0.219)	(0.209)	(<0.001)	(<0.001)
Block 1	0.136	0.136	0.126	0.170	0.168	0.167	0.142	0.137	0.169	0.166	0.114	0.105
	(<0.001)	(<0.001)	(<0.001)	(0.002)	(0.149)	(0.142)	(<0.001)	(<0.001)	(0.365)	(0.301)	(<0.001)	(<0.001)
Block SIM 1	0.135	0.134	0.138	0.169	0.167	0.167	0.140	0.135	0.168	0.167	0.112	0.104
	(<0.001)	(<0.001)	(0.002)	(0.002)	(0.131)	(0.120)	(<0.001)	(<0.001)	(0.336)	(0.349)	(<0.001)	(<0.001)
Block 2	0.132	0.131	0.121	0.167	0.166	0.165	0.139	0.134	0.167	0.165	0.110	0.100
	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(0.088)	(0.084)	(<0.001)	(<0.001)	(0.281)	(0.239)	(<0.001)	(<0.001)
Block SIM 2	0.132	0.129	0.129	0.166	0.165	0.165	0.137	0.132	0.168	0.167	0.108	0.100
	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(0.075)	(0.068)	(<0.001)	(<0.001)	(0.320)	(0.342)	(<0.001)	(<0.001)
Value-weighted	0.156	0.161	0.334	0.155	0.155	0.155	0.155	0.155	0.155	0.155	0.155	0.155
	(0.008)	(0.039)	(<0.001)	(<0.001)	(0.002)	(0.002)	(0.007)	(0.015)	(0.006)	(0.010)	(0.465)	(0.728)

Notes:
For each of the datasets listed in Table 2, this table reports the ex-post standard deviations of the portfolio strategies listed in Table 1. In the portfolio strategies 'Block 1', 'Block SIM 1', 'Block 2' and 'Block SIM 2', a two-block covariance matrix is used to construct the GMVP. In parentheses is the p-value of the difference between the standard deviation of each strategy and that of the $1/N$ portfolio, which is computed using an F-test. The standard deviations are annualized through multiplication by $\sqrt{12}$.

(v) Comparison between our Method and Strategies that Allow for GMVPs with Short Positions

One major objective of the method developed in this paper is to generate only long-only GMVPs. Nonetheless, as a final step, we also compare its performance with that of strategies from the existing literature that allow for GMVPs with short sale positions. We describe these strategies briefly in Table 7. We test their performance across all the datasets described in Sub-section 3(*ii*), using the methodology and performance criteria presented in Sub-section 3(*iii*).

Table 8 gives the ex-post Sharpe ratio of each portfolio strategy presented in Table 7, in comparison with the ex-post Sharpe ratio of the portfolio strategy 'Block SIM 2', which is one of the portfolio strategies we develop based on our new approach. We see that the 'Block SIM 2' strategy attains a higher ex-post Sharpe ratio than the strategies that allow for short positions in 68% of the cases. Moreover, in all cases where the strategies that allow for short positions yield negative Sharpe ratios, the *p*-value of

Table 7

List of Portfolio Strategies from the Literature that Allow for Short Sale Positions

No.	*Abbreviation*	*Portfolio Strategy*
1	CC1	GMVP - the covariance matrix is constructed using the constant correlation model (Elton and Gruber, 1973).
2	CC2	GMVP - the covariance matrix is constructed using a fifty-fifty combination of the constant correlation model and the sample pair-wise correlations (Elton et al., 2006).
3	1-Factor	GMVP - the covariance matrix is constructed using Sharpe's (1963) single-index model (Chan et al., 1999).
4	3-Factor	GMVP - the covariance matrix is constructed using the three-factor model of Fama and French (1993) (Chan et al., 1999).
5	4-Factor	GMVP - the covariance matrix is constructed using a four-factor model that includes, along with the three Fama and French Factors, a momentum factor (Chan et al., 1999).
6	LWID	GMVP - the covariance matrix is constructed by shrinking the sample covariance matrix towards the identity matrix (Ledoit and Wolf, 2004a).
7	LW1F	GMVP - the covariance matrix is constructed by shrinking the sample covariance matrix towards a one-factor matrix (Ledoit and Wolf, 2003).
8	LWCC	GMVP - the covariance matrix is constructed by shrinking the sample covariance matrix towards the matrix that is based on the constant correlation model (Ledoit and Wolf, 2004b).
9	NC1V	A one-norm constrained GMVP from DeMiguel et al. (2009b).
10	NC2V	A two-norm constrained GMVP from DeMiguel et al. (2009b).

Notes:
This table lists various portfolio strategies from the existing literature that allow for GMVPs with short sale positions and which we consider in the empirical analysis of Sub-section 4(v). The second column of the table gives the abbreviation used in referring to the strategy in Tables 8–10. Chan et al. (1999) report that the various factor models they examine generally provide similar results in terms of GMVP performance. Therefore, in our analysis, we single out only three of the factor models that Chan et al. (1999) evaluate. DeMiguel et al. (2009b) present several norm-constrained GMVPs. None of them outperforms the others across all the datasets and performance criteria that DeMiguel et al. (2009b) use in their empirical study. As 'NC1V' and 'NC2V' are two strategies that perform well in DeMiguel et al. (2009b), we choose to focus on them in our analysis.

Table 8

Sharpe Ratios for the Strategies from the Literature that Allow for Short Sale Positions

Portfolio	Stock Datasets			Portfolio Datasets								
Strategy	*Random 500*	*Random 250*	*International*	*48 Industries*	*90*	*90 + 1*	*90 + 3*	*90 + 4*	*20*	*20 + 1*	*20 + 3*	*20 + 4*
Block SIM 2	0.63	0.63	0.76	0.50	0.51	0.50	0.45	0.47	0.49	0.46	0.32	0.36
CC1	0.51	0.62	0.83	0.70	0.52	0.47	0.04	0.09	0.52	0.37	<0.00	<0.00
	(0.555)	(0.993)	(0.770)	(0.367)	(0.943)	(0.892)	(0.106)	(0.166)	(0.831)	(0.650)	(0.017)	(0.038)
CC2	0.49	0.61	0.69	0.52	0.65	0.55	<0.00	<0.00	0.63	0.35	<0.00	<0.00
	(0.306)	(0.890)	(0.840)	(0.935)	(0.276)	(0.664)	(<0.001)	(0.002)	(0.243)	(0.389)	(<0.001)	(<0.001)
1-Factor	0.68	0.75	0.73	0.73	0.61	<0.00	<0.00	<0.00	0.61	<0.00	<0.00	<0.00
	(0.698)	(0.368)	(0.910)	(0.315)	(0.568)	(0.001)	(<0.001)	(0.001)	(0.467)	(<0.001)	(<0.001)	(<0.001)
3-Factor	0.55	0.67	0.60	0.69	0.66	<0.00	<0.00	<0.00	0.63	<0.00	<0.00	<0.00
	(0.458)	(0.712)	(0.242)	(0.305)	(0.187)	(<0.001)	(<0.001)	(<0.001)	(0.231)	(<0.001)	(<0.001)	(<0.001)
4-Factor	0.55	0.66	0.60	0.64	0.65	<0.00	<0.00	<0.00	0.64	<0.00	<0.00	<0.00
	(0.423)	(0.734)	(0.262)	(0.446)	(0.213)	(<0.001)	(<0.001)	(<0.001)	(0.195)	(<0.001)	(<0.001)	(<0.001)
LWID	0.62	0.64	0.56	0.49	0.62	0.57	<0.00	0.07	0.64	0.44	<0.00	<0.00
	(0.878)	(0.888)	(0.329)	(0.944)	(0.340)	(0.515)	(<0.001)	(0.007)	(0.148)	(0.867)	(<0.001)	(<0.001)
LW1F	0.55	0.64	0.62	0.59	0.61	0.46	<0.00	<0.00	0.74	0.21	<0.00	<0.00
	(0.419)	(0.892)	(0.446)	(0.679)	(0.410)	(0.775)	(<0.001)	(<0.001)	(0.047)	(0.075)	(<0.001)	(<0.001)
LWCC	0.53	0.60	0.68	0.47	0.50	0.44	<0.00	<0.00	0.51	0.34	<0.00	<0.00
	(0.508)	(0.862)	(0.835)	(0.836)	(0.962)	(0.743)	(<0.001)	(0.002)	(0.878)	(0.492)	(<0.001)	(<0.001)
NC1V	0.61	0.64	0.52	0.55	0.56	0.52	<0.00	0.18	0.56	0.19	<0.00	<0.00
	(0.843)	(0.913)	(0.233)	(0.711)	(0.616)	(0.813)	(<0.001)	(0.131)	(0.351)	(0.010)	(<0.001)	(<0.001)
NC2V	0.63	0.65	0.50	0.37	0.64	0.42	<0.00	0.45	0.55	0.21	<0.00	<0.00
	(0.984)	(0.733)	(0.171)	(0.482)	(0.335)	(0.567)	(<0.001)	(0.924)	(0.550)	(0.024)	(<0.001)	(<0.001)

Notes:

For each of the datasets listed in Table 2, this table reports the ex-post Sharpe ratios of the portfolio strategies listed in Table 7 and of 'Block SIM 2', one of the strategies we develop in this paper, for which a two-block covariance matrix is used to construct the GMVP. In parentheses is the p-value of the difference between the Sharpe ratio of each strategy and that of the 'Block SIM 2' strategy, which is computed using the studentized time series bootstrap method introduced in Ledoit and Wolf (2008). The Sharpe ratios are annualized through multiplication by $\sqrt{12}$.

the difference between the Sharpe ratio of our method and that of each of these strategies is smaller than 5% (and in most cases even smaller than 0.1%). In almost all other cases, the differences between the Sharpe ratios are statistically insignificant at conventional levels.

Table 9 reports the results for portfolio turnover. In Panel A of Table 9, we can see that in 95% of the cases the raw turnover for our portfolio strategy 'Block SIM 2' is lower than that of the strategies from the literature that allow for short sale positions. Thus, the relatively good performance, in terms of Sharpe ratio, of our new portfolio approach compared to the methods that allow for short positions even improves when turnover is taken into account. The results presented in Panel B of Table 9 confirm the last conclusion. In virtually all cases, we obtain a positive return-loss, implying that, in virtually all cases, the 'Block SIM 2' strategy yields a higher Sharpe ratio than the strategies that allow for short positions after controlling for transactions costs. Further, the return-loss of these strategies compared to our approach is non-trivial, reaching an average of 6.6% in annual terms.

We notice that the strategies that allow for short sale positions indeed generate GMVPs with significant short positions. We define the amount of short positions as the sum of all negative portfolio weights, and calculate the average amount of short positions over all the GMVPs that are constructed based on each strategy in each dataset. The range we obtain for the average amount of short positions is from −13% to −403%, with an average of −124%.[18] These significant short sale positions of course induce a stronger diversification effect than in our approach, which is set to generate only long-only GMVPs. As a result, we can see in Table 10 that our method yields a higher ex-post standard deviation than the strategies that allow for short positions in 85% of the cases. In most of these cases, the differences between the standard deviation of each strategy and that of our method are statistically significant at the 1% level (with p-values that are mostly even smaller than 0.1%).

We believe that, taken together, the results of the comparison between our approach and the strategies from the literature that allow for GMVPs with short sale positions uphold the conclusion that the block structure for the covariance matrix could have an impact on practical portfolio choice.

5. CONCLUDING REMARKS

To the extent that short sales are considered an undesirable feature of portfolio optimization, there is an interest in discussing long-only efficient portfolios. Specifically, there is an interest in finding estimators of the covariance matrix that generate a long-only GMVP, as several empirical studies show that ex-post the GMVP performs at least as well as other frontier portfolios. Imposing short sale constraints on the optimization problem, no matter what covariance matrix estimator is used, enables us to obtain a long-only GMVP. However, at least from a theoretical point of view, this is problematic, since the weights of the GMVP can only be found numerically and not analytically. Imposing the short sale constraints also generates 'corner' solutions with zero weights in many assets.

18 An average amount of short positions of –124% means that on average for every dollar invested in the portfolio we short \$1.24 worth of stocks, while buying \$2.24 worth of other stocks. To save space, in the paper, we do not report the average amount of short positions for each strategy in each dataset. This information is available upon request from the authors.

Table 9

Portfolio Turnovers for the Strategies from the Literature that Allow for Short Sale Positions

	Stock Datasets			*Portfolio Datasets*								
Portfolio Strategy	*Random 500*	*Random 250*	*International*	*48 Industries*	*90*	*90 + 1*	*90 + 3*	*90 + 4*	*20*	*20 + 1*	*20 + 3*	*20 + 4*
Panel A: Turnover of each Portfolio Strategy												
Block SIM 2	0.047	0.046	0.059	0.047	0.051	0.049	0.041	0.040	0.048	0.044	0.036	0.040
CC1	0.146	0.142	0.148	0.089	0.120	0.121	0.090	0.085	0.092	0.091	0.047	0.046
CC2	0.197	0.186	0.204	0.121	0.177	0.180	0.097	0.093	0.093	0.107	0.048	0.042
1-Factor	0.116	0.115	0.099	0.079	0.117	0.451	0.167	0.130	0.090	0.272	0.064	0.055
3-Factor	0.132	0.130	0.107	0.117	0.167	0.504	0.084	0.076	0.133	0.425	0.054	0.050
4-Factor	0.135	0.132	0.108	0.123	0.174	0.469	0.079	0.082	0.144	0.379	0.050	0.053
LWID	0.160	0.161	0.162	0.130	0.214	0.211	0.084	0.078	0.083	0.090	0.039	0.034
LW1F	0.146	0.136	0.166	0.165	0.248	0.268	0.089	0.080	0.220	0.247	0.048	0.043
LWCC	0.174	0.157	0.202	0.142	0.142	0.143	0.097	0.094	0.095	0.098	0.050	0.044
NC1V	0.123	0.126	0.159	0.042	0.064	0.065	0.063	0.067	0.039	0.043	0.043	0.040
NC2V	0.150	0.112	0.168	0.271	0.334	0.311	0.070	0.092	0.067	0.079	0.044	0.040
Panel B: Return-Loss Relative to the Portfolio Strategy 'Block SIM 2'												
CC1	0.056	0.060	−0.002	0.031	0.089	0.079	0.054	0.049	0.052	0.027	0.059	0.069
CC2	0.079	0.062	0.051	0.055	0.098	0.067	0.055	0.065	0.050	0.033	0.049	0.052
1-Factor	0.054	0.059	0.032	0.035	0.085	0.264	0.147	0.119	0.034	0.138	0.046	0.082
3-Factor	0.061	0.060	0.041	0.052	0.084	0.227	0.071	0.087	0.062	0.164	0.029	0.053
4-Factor	0.062	0.059	0.039	0.057	0.086	0.183	0.065	0.093	0.063	0.132	0.030	0.053
LWID	0.056	0.052	0.054	0.062	0.090	0.066	0.048	0.059	0.043	0.032	0.036	0.037
LW1F	0.063	0.056	0.061	0.069	0.119	0.081	0.054	0.061	0.094	0.086	0.038	0.044
LWCC	0.064	0.051	0.050	0.062	0.082	0.071	0.058	0.069	0.054	0.026	0.050	0.050
NC1V	0.059	0.059	0.060	0.050	0.071	0.079	0.057	0.062	0.035	0.037	0.028	0.038
NC2V	0.055	0.046	0.058	0.132	0.129	0.063	0.060	0.060	0.049	0.039	0.029	0.038

Notes:

For each of the datasets listed in Table 2, Panel A reports the turnovers of the portfolio strategies listed in Table 7 and of 'Block SIM 2', one of the strategies we develop in this paper, for which a two-block covariance matrix is used to construct the GMVP. Panel B reports the return-loss, which is the extra return a strategy needs to provide for its Sharpe ratio to equal that of the 'Block SIM 2' strategy in the presence of proportional transactions costs of 50 basis points. Note that a positive return-loss is obtained when the Sharpe ratio of a portfolio strategy is lower than that of the 'Block SIM 2' strategy in the presence of the proportional transactions costs. The return-losses are annualized through multiplication by 12.

Table 10

Volatility for the Strategies from the Literature that Allow for Short sale Positions

Portfolio Strategy	*Stock Datasets*			*Portfolio Datasets*								
	Random 500	*Random 250*	*International*	*48 Industries*	*90*	*90 + 1*	*90 + 3*	*90 + 4*	*20*	*20 + 1*	*20 + 3*	*20 + 4*
Block SIM 2	0.132	0.129	0.129	0.166	0.165	0.165	0.137	0.132	0.168	0.167	0.108	0.100
CC1	0.129	0.127	0.122	0.134	0.185	0.182	0.129	0.123	0.184	0.178	0.126	0.118
	(0.657)	(0.721)	(0.466)	(<0.001)	(0.015)	(0.030)	(0.175)	(0.117)	(0.053)	(0.163)	(<0.001)	(<0.001)
CC2	0.102	0.105	0.113	0.124	0.148	0.145	0.050	0.050	0.148	0.140	0.037	0.041
	(<0.001)	(<0.001)	(0.091)	(<0.001)	(0.013)	(0.005)	(<0.001)	(<0.001)	(0.007)	(<0.001)	(<0.001)	(<0.001)
1-Factor	0.110	0.110	0.121	0.128	0.173	0.197	0.126	0.117	0.175	0.184	0.087	0.078
	(<0.001)	(<0.001)	(0.424)	(<0.001)	(0.349)	(<0.001)	(0.066)	(0.006)	(0.348)	(0.031)	(<0.001)	(<0.001)
3-Factor	0.105	0.106	0.124	0.128	0.146	0.146	0.027	0.027	0.151	0.149	0.030	0.030
	(<0.001)	(<0.001)	(0.600)	(<0.001)	(0.006)	(0.010)	(<0.001)	(<0.001)	(0.017)	(0.015)	(<0.001)	(<0.001)
4-Factor	0.104	0.105	0.124	0.126	0.146	0.144	0.021	0.024	0.150	0.145	0.023	0.027
	(<0.001)	(<0.001)	(0.599)	(<0.001)	(0.007)	(0.003)	(<0.001)	(<0.001)	(0.012)	(0.003)	(<0.001)	(<0.001)
LWID	0.112	0.113	0.131	0.129	0.154	0.152	0.044	0.043	0.147	0.141	0.024	0.029
	(<0.001)	(0.003)	(0.828)	(<0.001)	(0.119)	(0.090)	(<0.001)	(<0.001)	(0.003)	(<0.001)	(<0.001)	(<0.001)
LW1F	0.102	0.102	0.114	0.132	0.155	0.151	0.043	0.042	0.151	0.143	0.021	0.024
	(<0.001)	(<0.001)	(0.119)	(<0.001)	(0.152)	(0.066)	(<0.001)	(<0.001)	(0.021)	(<0.001)	(<0.001)	(<0.001)
LWCC	0.106	0.109	0.112	0.126	0.184	0.180	0.072	0.062	0.183	0.175	0.032	0.027
	(<0.001)	(<0.001)	(0.088)	(<0.001)	(0.022)	(0.050)	(<0.001)	(<0.001)	(0.072)	(0.282)	(<0.001)	(<0.001)
NC1V	0.116	0.118	0.135	0.126	0.149	0.147	0.033	0.047	0.147	0.139	0.031	0.035
	(0.006)	(0.046)	(0.559)	(<0.001)	(0.019)	(0.012)	(<0.001)	(<0.001)	(0.003)	(<0.001)	(<0.001)	(<0.001)
NC2V	0.114	0.117	0.131	0.153	0.166	0.156	0.032	0.057	0.151	0.134	0.031	0.035
	(0.002)	(0.036)	(0.815)	(0.065)	(0.975)	(0.229)	(<0.001)	(<0.001)	(0.017)	(<0.001)	(<0.001)	(<0.001)

Notes:
For each of the datasets listed in Table 2, this table reports the ex-post standard deviations of the portfolio strategies listed in Table 7 and of 'Block SIM 2', one of the strategies we develop in this paper, for which a two-block covariance matrix is used to construct the GMVP. In parentheses is the p-value of the difference between the standard deviation of each strategy and that of the 'Block SIM 2' strategy, which is computed using an F-test. The standard deviations are annualized through multiplication by $\sqrt{12}$.

In this paper, we introduce a new structure for the covariance matrix — the block structure — which under simple and directly computable conditions generates (in an unconstrained optimization) a long-only GMVP whose weights can be found analytically. With these conditions, 'corner' solutions with many zero weights are not necessarily obtained.

We construct a block covariance matrix, by dividing the portfolio's stocks into several groups (blocks). Within each block, the covariance between stocks is identical for all pairs of stocks in the block. The covariance between stocks from different blocks is also identical for all pairs. We show that the weights of a GMVP constructed using the block matrix can be found analytically. These weights are positive as long as the variance of each stock is greater than both its within-block covariance and its between-block covariance, and as long as the within-block covariances are not smaller than the between-block covariance.

We conduct an empirical analysis and show that portfolio strategies based on a GMVP that is constructed using an easy to calculate block covariance matrix perform well. In particular, they generally outperform the $1/N$ portfolio which is in the focus of recent papers which show that existing and quite advanced mean-variance theory-based portfolio strategies do not consistently outperform the $1/N$ portfolio. Thus, our findings may suggest that our new approach, and not the $1/N$ portfolio, should serve as a benchmark to assess the performance of portfolio rules, and that the block structure for the covariance matrix of asset returns could have an impact on practical portfolio choice.

APPENDIX A

The General Block Covariance Matrix

We assume a universe with n stocks. A covariance matrix $\mathbf{\Sigma}$ is said to be a *block matrix* if it has the following form:

$$
\Sigma = \left(\begin{array}{cccccccccccccccc}
s_1^2 & \eta_1 & \cdots & \eta_1 & & & & & & & & & & & & \\
\eta_1 & \ddots & \ddots & \vdots & & & & \eta & & & & & & \eta & & \\
\vdots & \ddots & \ddots & \eta_1 & & & & & & & & & & & & \\
\eta_1 & \cdots & \eta_1 & s_k^2 & & & & & & & & & & & & \\
 & & & & s_{k+1}^2 & \eta_2 & \cdots & & & \eta_2 & & & & & & \\
 & & & & \eta_2 & \ddots & \ddots & & & \vdots & & & & & & \\
 & & & & \vdots & \ddots & s_l^2 & & & & & & & & & \\
 & \eta & & & & & & \ddots & & & & & & \eta & & \\
 & & & & & & & & & \eta_2 & & & & & & \\
 & & & & \eta_2 & \cdots & & & \eta_2 & s_p^2 & & & & & & \\
 & & & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & & s_r^2 & \eta_M & \cdots & & \eta_M \\
 & & & & & & & & & & & \eta_M & \ddots & & & \\
 & \eta & & & & & & \eta & & & & \vdots & \ddots & s_t^2 & & \\
 & & & & & & & & & & & & & & \ddots & \eta_M \\
 & & & & & & & & & & & \eta_M & & & \eta_M & s_n^2
\end{array}\right)
$$

Here we have M blocks of stocks (which are not required to include equal numbers of stocks), s_j^2, $j = 1, \ldots, n$, are the variances, η_i, $i = 1, \ldots, M$, are the within-block covariances, and η is the between-block covariance.

Proposition 2 below characterizes sufficient conditions on η_i and η under which the block matrix produces a long-only GMVP (in an unconstrained optimization). Here we present only the simple and directly computable conditions that are obtained when all the covariances are restricted to be nonnegative. As in the special case of the two-block matrix, for the general block matrix too the sufficient conditions are: (i) the variance of each stock is greater than both its within-block covariance and its between-block covariance, and (ii) the within-block covariances are not smaller than the between-block covariance.

Proposition 2: Suppose that $\mathbf{\Sigma}$ is a block matrix and the covariances $\eta_i, i = 1, \ldots M$ and η are nonnegative. Then $\mathbf{\Sigma}$ produces a long-only GMVP, if the following conditions on η_i and η hold:

$$\begin{aligned} &0 \leq \eta_i < \mathrm{minvar}_i, \quad i = 1, \ldots, M \\ &0 \leq \eta \leq \min(\eta_i), \quad i = 1, \ldots, M, \end{aligned}$$

where minvar_i denotes the minimal variance in block i. The proof for Proposition 2 is available upon request from the authors.

It can be shown that the expression for the weight in the GMVP of stock j from block i ($i = 1, \ldots, M$; $j = 1, \ldots, Z_i$, where Z_i denotes the number of stocks in block i) is:

$$w_j = \frac{1}{s_j^2 - \eta_i} \cdot \frac{1 + \sum\limits_{k=1;k\neq i}^{M} B_k + \sum\limits_{k,l=1;k<l;k,l\neq i}^{M} B_k B_l + \sum\limits_{k,l,r=1;k<l<r;k,l,r\neq i}^{M} B_k B_l B_r + \cdots + \underset{k<l<r<\ldots t\ldots<M;k,l,r\ldots,t\ldots,M\neq i}{B_k B_l B_r \ldots B_t \ldots B_M}}{\sum\limits_{i=1}^{M} A_i C_i},$$

where $A_i = \sum_{j=1}^{Z_i} \frac{1}{s_j^2 - \eta_i}$, $B_i = (\eta_i - \eta) A_i$, $i = 1, \ldots, M$,
and $C_i = 1 + \sum_{k=1;k\neq i}^{M} B_k + \sum_{k,l=1;k<l;k,l\neq i}^{M} B_k B_l + \sum_{k,l,r=1;k<l<r;k,l,r\neq i}^{M} B_k B_l B_r + \cdots + \underset{k<l<r<\ldots t\ldots<M;k,l,r\ldots,t\ldots,M\neq i}{B_k B_l B_r \ldots B_t \ldots B_M}$.

One can see that when $M = 2$, we indeed get the expression for the GMVP's weights that is presented in Section 2, where we discuss the two-block covariance matrix case.

APPENDIX B

Proof for Proposition 1

The proof consists of two phases:

1. Finding sufficient conditions on η_1, η_2 and η for which a long-only GMVP is obtained.
2. Modifying the conditions to guarantee that the two-block matrix is an invertible covariance matrix. Namely, showing that the eigenvalues of the matrix are strictly

positive, since every positive definite symmetric matrix is an invertible covariance matrix.

Phase 1—long-only GMVP

Denote the first column of $\mathbf{\Sigma}^{-1}$ by:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_j \\ x_{j+1} \\ \vdots \\ x_n \end{pmatrix},$$

where $\mathbf{\Sigma}^{-1}$ denotes the inverse matrix of $\mathbf{\Sigma}$.

Then, since $\mathbf{\Sigma}\Sigma^{-1} = \mathbf{I}$:

$$\mathbf{\Sigma} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_j \\ x_{j+1} \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Given the structure of $\mathbf{\Sigma}$, writing this explicitly yields:

$$\begin{aligned} & s_1^2 x_1 + \eta_1 x_2 + \cdots + \eta_1 x_j + \eta x_{j+1} + \eta x_{j+2} + \cdots + \eta x_n = 1 \\ & \eta_1 x_1 + s_2^2 x_2 + \cdots + \eta_1 x_j + \eta x_{j+1} + \eta x_{j+2} + \cdots + \eta x_n = 0 \\ & \vdots \\ & \eta_1 x_1 + \eta_1 x_2 + \cdots + s_j^2 x_j + \eta x_{j+1} + \eta x_{j+2} + \cdots + \eta x_n = 0 \\ & \eta x_1 + \eta x_2 + \cdots + \eta x_j + s_{j+1}^2 x_{j+1} + \eta_2 x_{j+2} + \cdots + \eta_2 x_n = 0 \\ & \eta x_1 + \eta x_2 + \cdots + \eta x_j + \eta_2 x_{j+1} + s_{j+2}^2 x_{j+2} + \cdots + \eta_2 x_n = 0 \\ & \vdots \\ & \eta x_1 + \eta x_2 + \cdots + \eta x_j + \eta_2 x_{j+1} + \eta_2 x_{j+2} + \cdots + s_n^2 x_n = 0, \end{aligned} \tag{A1}$$

and therefore:

$$
\begin{aligned}
&\left(s_1^2-\eta_1\right)x_1+\eta_1\sum_{i=1}^{j}x_i+\eta\sum_{i=j+1}^{n}x_i=1\\
&\left(s_2^2-\eta_1\right)x_2+\eta_1\sum_{i=1}^{j}x_i+\eta\sum_{i=j+1}^{n}x_i=0\\
&\vdots\\
&\left(s_j^2-\eta_1\right)x_j+\eta_1\sum_{i=1}^{j}x_i+\eta\sum_{i=j+1}^{n}x_i=0\\
&\eta\sum_{i=1}^{j}x_i+\left(s_{j+1}^2-\eta_2\right)x_{j+1}+\eta_2\sum_{i=j+1}^{n}x_i=0\\
&\eta\sum_{i=1}^{j}x_i+\left(s_{j+2}^2-\eta_2\right)x_{j+2}+\eta_2\sum_{i=j+1}^{n}x_i=0\\
&\vdots\\
&\eta\sum_{i=1}^{j}x_i+\left(s_n^2-\eta_2\right)x_n+\eta_2\sum_{i=j+1}^{n}x_i=0.
\end{aligned}
$$

Dividing the first j equations by $s_i^2-\eta_1$ (assuming $\eta_1 \neq s_i^2, i=1,\ldots,j$) and dividing the last n-j equations by $s_i^2-\eta_2$ (assuming $\eta_2 \neq s_i^2, i=j+1,\ldots,n$) gives:

$$
\begin{aligned}
&x_1+\frac{\eta_1}{s_1^2-\eta_1}\sum_{i=1}^{j}x_i+\frac{\eta}{s_1^2-\eta_1}\sum_{i=j+1}^{n}x_i=\frac{1}{s_1^2-\eta_1}\\
&\vdots\\
&x_j+\frac{\eta_1}{s_j^2-\eta_1}\sum_{i=1}^{j}x_i+\frac{\eta}{s_j^2-\eta_1}\sum_{i=j+1}^{n}x_i=0\\
&x_{j+1}+\frac{\eta}{s_{j+1}^2-\eta_2}\sum_{i=1}^{j}x_i+\frac{\eta_2}{s_{j+1}^2-\eta_2}\sum_{i=j+1}^{n}x_i=0\\
&\vdots\\
&x_n+\frac{\eta}{s_n^2-\eta_2}\sum_{i=1}^{j}x_i+\frac{\eta_2}{s_n^2-\eta_2}\sum_{i=j+1}^{n}x_i=0.
\end{aligned}
\tag{A2}
$$

Summing the first j equations in (A2) and rearranging terms gives:

$$
\left(1+\eta_1\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}\right)\sum_{i=1}^{j}x_i+\eta\sum_{i=1}^{j}\frac{1}{s_i^2-\eta_1}\sum_{i=j+1}^{n}x_i=\frac{1}{s_1^2-\eta_1}. \tag{A3}
$$

Summing the last *n-j* equations in (A2) and rearranging terms gives:

$$\sum_{i=j+1}^{n} x_i = -\frac{\eta \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}} \sum_{i=1}^{j} x_i. \tag{A4}$$

And now substituting (A4) into (A3) and rearranging terms gives:

$$\sum_{i=1}^{j} x_i = \frac{1}{s_1^2 - \eta_1} \cdot \frac{1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\left(1 + \eta_1 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}\right)\left(1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}\right) - \eta^2 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}.$$

Denote:

$$\Delta = \left(1 + \eta_1 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}\right)\left(1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}\right) - \eta^2 \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2},$$

and therefore:

$$\sum_{i=1}^{j} x_i = \frac{1}{s_1^2 - \eta_1} \cdot \frac{1 + \eta_2 \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}. \tag{A5}$$

Substituting (A5) into (A4) gives:

$$\sum_{i=j+1}^{n} x_i = \frac{1}{s_1^2 - \eta_1} \cdot \frac{-\eta \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}. \tag{A6}$$

Adding equations (A5) and (A6) gives the sum of (the elements in) the first column of $\mathbf{\Sigma}^{-1}$. Since $\mathbf{\Sigma}^{-1}$ is symmetric (because $\mathbf{\Sigma}$ is symmetric) this is also the sum of (the elements in) the first row of $\mathbf{\Sigma}^{-1}$:

$$\sum_{j=1}^{n} \mathbf{\Sigma}_{1j}^{-1} = \frac{1}{s_1^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta) \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}. \tag{A7}$$

Generalizing the last result and finding the sum of each one of the rows of $\mathbf{\Sigma}^{-1}$ is done as follows: We start with the first j rows of $\mathbf{\Sigma}^{-1}$. We repeat the above procedure [i.e., deriving equations (A1)-(A7)] j times. Each time the only difference is that the only row in (A1) that equals 1 moves one place ahead. For example, for the sum of the second row of $\mathbf{\Sigma}^{-1}$, the second row in (A1), and not the first row, equals 1. Thus, the general expression for the sum of one of the first j rows of $\boldsymbol{\Sigma}^{-1}$ is:

$$\sum_{j=1}^{n} \boldsymbol{\Sigma}_{ij}^{-1} = \frac{1}{s_i^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta) \displaystyle\sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}. \tag{A8}$$

Repeating a similar procedure n-j times and applying 'symmetric considerations' enables us to find the general expression for the sum of one of the last n-j rows of $\boldsymbol{\Sigma}^{-1}$:

$$\sum_{j=1}^{n} \boldsymbol{\Sigma}_{ij}^{-1} = \frac{1}{s_i^2 - \eta_2} \cdot \frac{1 + (\eta_1 - \eta) \displaystyle\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}{\Delta}. \tag{A9}$$

The vector of the GMVP weights is $\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}$. For the weight of stock i, in the numerator we have the sum of row i in $\boldsymbol{\Sigma}^{-1}$ and in the denominator we have the sum of (all the elements in) $\boldsymbol{\Sigma}^{-1}$. Hence, in order to obtain the denominator, we use (A8) and (A9):

$$\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1} = \sum_{i=1}^{j}\left(\frac{1}{s_i^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta) \displaystyle\sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}\right) + \sum_{i=j+1}^{n}\left(\frac{1}{s_i^2 - \eta_2} \cdot \frac{1 + (\eta_1 - \eta) \displaystyle\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}}{\Delta}\right),$$

and after rearranging a bit more we get:

$$\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1} = \frac{\displaystyle\sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} + \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2} + (\eta_1 + \eta_2 - 2\eta) \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1} \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}}{\Delta}. \tag{A10}$$

Substituting (A8), (A9) and (A10) into $\mathbf{w} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^{\mathbf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{1}}$ gives the expressions for the weight in the GMVP of a stock from the first and the second block respectively:

$$\begin{aligned} w_i &= \frac{1}{s_i^2 - \eta_1} \cdot \frac{1 + (\eta_2 - \eta)\, A_2}{A_1 + A_2 + (\eta_1 + \eta_2 - 2\eta) A_1 A_2}, \quad i = 1, \ldots, j, \\ w_i &= \frac{1}{s_i^2 - \eta_2} \cdot \frac{1 + (\eta_1 - \eta)\, A_1}{A_1 + A_2 + (\eta_1 + \eta_2 - 2\eta) A_1 A_2}, \quad i = j+1, \ldots, n. \end{aligned} \tag{A11}$$

where: $A_1 = \sum_{i=1}^{j} \frac{1}{s_i^2 - \eta_1}$, $A_2 = \sum_{i=j+1}^{n} \frac{1}{s_i^2 - \eta_2}$.

And now we can easily see that the vector **w** is strictly positive if the following set of conditions holds:

$$\begin{aligned} \eta_1 &< \min\left(s_i^2\right), \quad i = 1, \ldots, j \\ \eta_2 &< \min\left(s_i^2\right), \quad i = j+1, \ldots, n. \\ \eta &\leq \min(\eta_1, \eta_2). \end{aligned} \tag{A12}$$

Phase 2—strictly positive eigenvalues

$\boldsymbol{\Sigma}$ is a symmetric real matrix. Therefore, it has real eigenvalues and it can be diagonalized:

$$\mathbf{R}^{-1}\boldsymbol{\Sigma}R = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix},$$

where $\mathbf{R}$ denotes the diagonalizing matrix, $\mathbf{R}^{-1}$ denotes the inverse matrix of $\mathbf{R}$, and$\boldsymbol{\Lambda}$ denotes the diagonal matrix, whose diagonal elements $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{\Sigma}$.

Hence, $\boldsymbol{\Sigma}R = R\Lambda$ and the element ij of $\boldsymbol{\Sigma}R$ is given by:

$$(\boldsymbol{\Sigma}R)_{ij} = \sum_{k=1}^{n} \boldsymbol{\Sigma}_{ik}\mathbf{R}_{kj} = \lambda_j \mathbf{R}_{ij}.$$

This is a set of equations that determines the elements of column j in $\mathbf{R}$. It is a set that fits any column in $\mathbf{R}$, and for the sake of convenience we omit the index j from the last expression, to obtain:

$$\sum_{k=1}^{n} \boldsymbol{\Sigma}_{ik}\mathbf{R}_k = \lambda \mathbf{R}_i.$$

Now, substituting the elements of $\boldsymbol{\Sigma}$ into the last expression gives:

$$\begin{aligned} s_i^2\mathbf{R}_i + \eta_1 \sum_{k=1, k\neq i}^{j} \mathbf{R}_k + \eta \sum_{k=j+1}^{n} \mathbf{R}_k &= \lambda \mathbf{R}_i, \quad i = 1, \ldots, j \\ s_i^2\mathbf{R}_i + \eta \sum_{k=1}^{j} \mathbf{R}_k + \eta_2 \sum_{k=j+1, k\neq i}^{n} \mathbf{R}_k &= \lambda \mathbf{R}_i, \quad i = j+1, \ldots, n. \end{aligned}$$

Because $\mathbf{R}_i$ exists $\forall i$, we can divide the two expressions by $\dfrac{1}{\lambda-(s_i^2-\eta_1)}$ and $\dfrac{1}{\lambda-(s_i^2-\eta_2)}$ respectively. Therefore, after rearranging we obtain:

$$\mathbf{R}_i=\frac{\eta_1}{\lambda-\left(s_i^2-\eta_1\right)}\sum_{k=1}^{j}\mathbf{R}_k+\frac{\eta}{\lambda-\left(s_i^2-\eta_1\right)}\sum_{k=j+1}^{n}\mathbf{R}_k=0,\quad i=1,\ldots,j$$

$$\mathbf{R}_i=\frac{\eta}{\lambda-\left(s_i^2-\eta_2\right)}\sum_{k=1}^{j}\mathbf{R}_k+\frac{\eta_2}{\lambda-\left(s_i^2-\eta_2\right)}\sum_{k=j+1}^{n}\mathbf{R}_k=0,\quad i=j+1,\ldots,n.$$

Summing the two expressions over all the possible values of i (and replacing the index i with k) gives:

$$\sum_{k=1}^{j}\mathbf{R}_k=\eta_1\sum_{k=1}^{j}\frac{1}{\lambda-\left(s_k^2-\eta_1\right)}\sum_{k=1}^{j}\mathbf{R}_k+\eta\sum_{k=1}^{j}\frac{1}{\lambda-\left(s_k^2-\eta_1\right)}\sum_{k=j+1}^{n}\mathbf{R}_k$$

$$\sum_{k=j+1}^{n}\mathbf{R}_k=\eta\sum_{k=j+1}^{n}\frac{1}{\lambda-\left(s_k^2-\eta_2\right)}\sum_{k=1}^{j}\mathbf{R}_k+\eta_2\sum_{k=j+1}^{n}\frac{1}{\lambda-\left(s_k^2-\eta_2\right)}\sum_{k=j+1}^{n}\mathbf{R}_k,$$

and after rearranging we get:

$$\left[\eta_1\sum_{k=1}^{j}\frac{1}{\lambda-\left(s_k^2-\eta_1\right)}-1\right]\sum_{k=1}^{j}\mathbf{R}_k+\eta\sum_{k=1}^{j}\frac{1}{\lambda-\left(s_k^2-\eta_1\right)}\sum_{k=j+1}^{n}\mathbf{R}_k=0$$

$$\eta\sum_{k=j+1}^{n}\frac{1}{\lambda-\left(s_k^2-\eta_2\right)}\sum_{k=1}^{j}\mathbf{R}_k+\left[\eta_2\sum_{k=j+1}^{n}\frac{1}{\lambda-\left(s_k^2-\eta_2\right)}-1\right]\sum_{k=j+1}^{n}\mathbf{R}_k=0.$$

These are two homogenous linear equations with two unknowns, $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$. It can be shown that $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$ cannot both equal zero. A set of homogenous linear equations has a solution other than the zero solution, if and only if its determinant equals 0. Therefore, and since $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$ exist, we get:[19]

$$\left[\eta_1\sum_{k=1}^{j}\frac{1}{\lambda-\left(s_k^2-\eta_1\right)}-1\right]\left[\eta_2\sum_{k=j+1}^{n}\frac{1}{\lambda-\left(s_k^2-\eta_2\right)}-1\right]=\eta^2\sum_{k=1}^{j}\frac{1}{\lambda-\left(s_k^2-\eta_1\right)}\times\sum_{k=j+1}^{n}\frac{1}{\lambda-\left(s_k^2-\eta_2\right)}.$$

Denote:

$$\mathrm{F}_1(\lambda)=\sum_{k=1}^{j}\frac{1}{\lambda-\left(s_k^2-\eta_1\right)}\quad\text{and}\quad\mathrm{F}_2(\lambda)=\sum_{k=j+1}^{n}\frac{1}{\lambda-\left(s_k^2-\eta_2\right)}.$$

Therefore, the last equation becomes:

$$\left[\eta_1\mathrm{F}_1(\lambda)-1\right]\left[\eta_2\mathrm{F}_2(\lambda)-1\right]=\eta^2\mathrm{F}_1(\lambda)\,\mathrm{F}_2(\lambda).$$

19 $\sum_{k=1}^{j}\mathbf{R}_k$ and $\sum_{k=j+1}^{n}\mathbf{R}_k$ exist because the diagonalizing procedure is well defined here.

$\boldsymbol{\Sigma}$ is finite. Thus, λ, $s_k^2 \forall k$, η_1 and η_2 are finite, and therefore $\mathrm{F}_i(\lambda) \neq 0, i = 1, 2$. Hence, we can divide the last expression by $\mathrm{F}_1(\lambda)\mathrm{F}_2(\lambda)$and obtain:

$$\left[\eta_1 - \frac{1}{\mathrm{F}_1(\lambda)}\right]\left[\eta_2 - \frac{1}{\mathrm{F}_2(\lambda)}\right] = \eta^2. \tag{A13}$$

It can be shown that for $\lambda \leq 0$:

$$-\infty < \frac{1}{\mathrm{F}_i(\lambda)} \leq -\frac{1}{\sum_i \frac{1}{s_k^2 - \eta_i}}, \quad i = 1, 2,$$

$$\text{where} \quad \Sigma_i = \begin{cases} \sum_{k=1}^{j}, & i = 1 \\ \sum_{k=j+1}^{n}, & i = 2 \end{cases}$$

and therefore:

$$\infty > \eta_i - \frac{1}{\mathrm{F}_i(\lambda)} \geq \eta_i + \frac{1}{\sum_i \frac{1}{s_k^2 - \eta_i}}, \quad i = 1, 2, \quad \lambda \leq 0. \tag{A14}$$

Now, we assume that:

$$\eta_1 > -\frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}}, \quad \eta_2 > -\frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}. \tag{A15}$$

Thus, from (A14) and (A15) we obtain that:

$$\left[\eta_1 - \frac{1}{\mathrm{F}_1(\lambda)}\right]\left[\eta_2 - \frac{1}{\mathrm{F}_2(\lambda)}\right] \geq \left(\eta_1 + \frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}\right), \quad \lambda \leq 0,$$

and if we also assume:

$$\left(\eta_1 + \frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}\right) > \eta^2, \tag{A16}$$

then:

$$\left[\eta_1 - \frac{1}{\mathrm{F}_1(\lambda)}\right]\left[\eta_2 - \frac{1}{\mathrm{F}_2(\lambda)}\right] > \eta^2, \quad \lambda \leq 0,$$

which means that under the conditions in (A15) and (A16) there are no nonpositive eigenvalues for which equation (A13) holds. Now, since equation (A13) holds under the conditions in (A15) and (A16), it means that under the conditions in (A15) and (A16) equation (A13) holds only for strictly positive eigenvalues. In other words, the conditions in (A15) and (A16) are sufficient to ensure that $\boldsymbol{\Sigma}$ is positive definite.[20]

Because of (A15), we can denote:

$$|\eta_{12}^*| = +\sqrt{\left(\eta_1 + \frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}}\right)\left(\eta_2 + \frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}\right)},$$

and write the set of the sufficient conditions from (A15) and (A16) again as follows:

$$\eta_1 > -\frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}}, \quad \eta_2 > -\frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}}, \quad -|\eta_{12}^*| < \eta < |\eta_{12}^*|.$$

It can be shown that:

$$\begin{aligned}
\eta_1 &> -\frac{1}{\sum_{k=1}^{j} \frac{1}{s_k^2 - \eta_1}} \quad \textit{iff} \quad -|\eta_1^*| < \eta_1 \le \min\left(s_k^2\right), \quad k = 1, \ldots, j \\
\eta_2 &> -\frac{1}{\sum_{k=j+1}^{n} \frac{1}{s_k^2 - \eta_2}} \quad \textit{iff} \quad -|\eta_2^*| < \eta_2 \le \min\left(s_k^2\right), \quad k = j+1, \ldots, n,
\end{aligned}$$

where $|\eta_1^*|$ and $|\eta_2^*|$ are respectively the unique solutions of the following equations:

$$|\eta_1^*| = 1 \Bigg/ \sum_{k=1}^{j} \frac{1}{s_k^2 + |\eta_1^*|} \quad \text{and} \quad |\eta_2^*| = 1 \Bigg/ \sum_{k=j+1}^{n} \frac{1}{s_k^2 + |\eta_2^*|}.$$

To sum up, $\boldsymbol{\Sigma}$ is positive definite if the following set of conditions holds (note that we replace the index k with i):

$$\begin{aligned}
&-|\eta_1^*| < \eta_1 \le \min\left(s_i^2\right), \quad i = 1, \ldots, j \\
&-|\eta_2^*| < \eta_2 \le \min\left(s_i^2\right), \quad i = j+1, \ldots, n. \\
&-|\eta_{12}^*| < \eta < |\eta_{12}^*|.
\end{aligned} \tag{A17}$$

Now, combining the sets of conditions from (A12) and (A17) gives the sufficient conditions of Proposition 1. Q.E.D.

20 It can be shown that these sufficient conditions are also necessary. However, we do not show this here, as our goal is to find a set of sufficient conditions on η_1, η_2 and η.

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