REGULAR COOPERATIVE BALANCED GAMES
WITH APPLICATIONS TO LINE-BALANCING

by

S. Anily*
M. Haviv†

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* The Leon Recanati Graduate School of Business Administration, Tel Aviv University, Ramat Aviv, Tel Aviv, 69978, Israel. Email: anily@post.tau.ac.il

† Department of Statistics and the Center for the Study of Rationality, Hebrew University of Jerusalem, 91905 Jerusalem, Israel. E-mail: moshe.haviv@gmail.com

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Regular cooperative balanced games with applications to line-balancing

Shoshana Anily*and Moshe Haviv†

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Abstract

The conventional definition of a cooperative game $G(N, V)$ with a set of players $N = \{1, \ldots, n\}$ and a characteristic function V, is quite rigid to alterations of the set of players N. Moreover, it may necessitate a large input of size that is exponential in n. However, the characteristic function of many games allows a simple, efficient and flexible presentation of the game. Here we deal with a set of games that we call *regular games*, which have a simple presentation: In regular games each player is characterized by a vector of quantitative properties, and the characteristic function value of a coalition depends only on the vectors of properties of its members. We show that some regular games in which players can cooperate with respect to some of their resources and whose immediate formulation does not fit the framework of market games, can nevertheless be transformed into the form of market games and hence they are totally balanced. In particular, they lead to a core allocation based on a competitive equilibrium prices of the transformed game.

1 Introduction

A general cooperative coalitional game is defined by a set of n players, $N = \{1, 2, \ldots, n\}$, that can break up into subsets. Any subset S of N, $\emptyset \subseteq S \subseteq N$, is called a *coalition*, where N itself is called the *grand-coalition*. Each coalition S is associated with a real non-negative value denoted by

*Faculty of Management, Tel Aviv University, Tel Aviv 69978, Israel. E-mail: anily@post.tau.ac.il

†Department of Statistics and the Center for the Study of Rationality, Hebrew University of Jerusalem, 91905 Jerusalem, Israel. E-mail: moshe.haviv@gmail.com.

$V(S)$, where $V(\emptyset) = 0$. The value $V(S)$ is the total cost inflicted on the members of coalition S if they cooperate. The function $V : 2^N \rightarrow \Re$ is called a *characteristic function*. The pair $G = (N, V)$ is said to be a *cooperative game with transferable utility*. The total cost incurred by all players of N depends on the partition of N into disjoint coalitions, so that if N is partitioned into m disjoint coalitions, $S_1 \cup S_2 \cup \ldots \cup S_m = N$, $1 \leq m \leq n$, then the total cost is $\sum_{i=1}^{m} V(S_i)$. This conventional definition of a cooperative game has the advantage of being general and simple, i.e., any coalitional game can be casted into this framework. Yet, its main drawback is its input size. Specifically, in order to fully describe a game $G = (N, V)$, $2^n - 1$ values are needed to be specified. The input size burden imposes a practical restriction on the size of games (number of players) that can be studied. For example, consider a *simple game*, where $V(S) \in \{0, 1\}$ and $V(N) = 1$. This game, as simple as it looks, necessitates the specification of $V(S)$ for any $S \subset N$. Thus, the generality of the common presentation of games comes at the cost of its limitations.

In this paper we discuss some structured coalitional games, especially games that arise in operations and service management, that allow a much more efficient presentation. We call the class of games that we focus on *Regular Games*: In a regular game each player is characterized by a vector of quantitative properties, called *vector of properties* and the cost of a coalition of size $m \geq 1$ is a function V_m of the m vectors of properties of its members, but it is otherwise independent of the identity of the players, or the number of players in the grand-coalition. Two consecutive functions V_m and V_{m+1} for $m \geq 0$ are linked through a simple relation. That means that all is needed in order to describe a regular game $G = (N, V)$ is the permissible domain of vectors of properties, the vector of properties of each player in N, a function V_m that maps m vectors of properties into $\Re$, and the linking relation that states V_{m+1} in terms of V_m,. Thus, the input size to describe such a game is of size $O(n)$. This proposed form of presentation for regular games, has the advantage of being flexible, meaning that the modifications required when changing the set of players N by adding, removing or duplicating some players are marginal: all is needed is an update of the collection of the vectors of properties of the new set of players.

Next we review the main concepts in cooperative games that are relevant to this paper. Given a game, the first question is whether the grand-coalition is the socially optimal formation of coalitions. A sufficient condition for that is the sub-additivity of its characteristic function: A game $G = (N, V)$ is called *subadditive* if for any two disjoint coalitions S and T, $V(S \cup T) \leq V(S) + V(T)$. Sub-additivity ensures that the socially best partition of the players of N to disjoint coalitions is when all players cooperate and join the grand-coalition. Subadditive games bear the concept of *economies of scope*, i.e., when each player, or set of players, contributes its own skills and resources, the total cost is no greater than the sum of the costs of the individual parts. On top of forming the grand-coalition, players of N need to establish a way that allocates the cost $V(N)$ among themselves, so that no group of players may resist this cooperation and decide to act alone. Several concepts of stability have been proposed in the literature. The most appealing is the *core*: A vector $x \in \Re^n$ is said to be *efficient* if $\sum_{i=1}^{n} x_i = V(N)$, and it is said to be a *core cost allocation* of the game if it is efficient and if $\sum_{i \in S} x_i \leq V(S)$ for any $S \subset N$.

The collection of all core allocations, called the *core* of the game, forms a simplex in $\Re^n$ as it is defined by a set of linear constraints with n decision variables. As the number of constraints that define the core is exponential in n, more specifically it is $2^n - 1$, finding a core allocation for a given game may be, in general, an intricate task. Indeed, this issue coupled with the possibility that the core is empty, makes the problem of finding a core allocation a real challenge in some games. Moreover, even if we can prove the non-emptiness of the core, the question of finding a cost allocation in the core may be non-trivial.

A cooperative game $G = (N, V)$ is said to be *balanced* if its core is non-empty, and *totally balanced* if its core and the cores of all its subgames are non-empty. Subadditivity is a necessary condition for total balancedness as if there exist disjoint coalitions S and T for which $V(S) + V(T) < V(S \cup T)$, the subgame $(S \cup T, V)$ has an empty core since any efficient allocation of $V(S \cup T)$ among the players of $S \cup T$ will be objected by at least one of the coalitions S or T. The literature provides two main conditions that are sufficient in order establish the total balancedness of a game.

- **Condition 1.** A game $G = (N, V)$ is a *concave game* if its characteristic function is concave, meaning that for any two coalitions $S, T \subseteq N$, $V(S \cup T) + V(S \cap T) \leq V(S) + V(T)$. Clearly, concave games are subadditive but not the other way around. It was shown in [15] that the core of a concave game possesses $n!$ extreme points, each of which being the vector of marginal contribution of the players to a different permutation of the players.

  Remark 1. A game is called an *Average Convex Game* if for any two disjoint coalitions $S, T \subset N$, the following inequality holds: $\sum_{i \in S}(V(S \cup T) - V(S \cup T \backslash \{i\})) \leq \sum_{i \in S}(V(S) - V(S \backslash \{i\}))$. This set of games includes as a proper subset the set of convex games. In [6] it is proved that the Shapley Value, see [15], of an average convex game is in the core of the game, proving that such games are also totally balanced.

- **Condition 2.** A *market game*, see e.g., Chapter 13 in [13], is defined as follows: Suppose there are ℓ types of inputs. An *input vector* is a non-negative vector in $\Re^\ell_+$. Each of the n players possesses an initial commitment vector $w_i \in \Re^\ell_+$, $1 \leq i \leq n$, which states a nonnegative quantity for each input. Moreover, each player is associated with a continuous and convex cost function $f_i : \Re^\ell_+ \rightarrow \Re_+$, $1 \leq i \leq n$. A profile $(z_i)_{i \in N}$ of input vectors for which $\sum_{i \in N} z_i = \sum_{i \in N} w_i$ is an *allocation.* The game is such that a coalition S of players looks for an optimal way to redistribute its members' commitments among its members in order to get a profile $(z_i)_{i \in S}$ of input vectors so as the sum of the costs across the members of S is minimized. Formally, for any $\emptyset \subseteq S \subseteq N$,

$$V(S) = \min \{ \sum_{i \in S} f_i(z_i) : z_i \in \Re^\ell_+, i \in S \text{ and } \sum_{i \in S} z_i = \sum_{i \in S} w_i \} \qquad (1)$$

  Remark 2. In [13] it is assumed that the functions $f_i(z_i)$, $1 \leq i \leq n$, are non-increasing but as noted in [4] page 163 footnote 2, this in fact is not required.

- **Condition 3.** A regular market game which is sub-additive and homogeneous of degree one as defined in [2]. Some preliminaries given in Section 2 are needed, so we elaborate on this condition in the sequel.

Market games are not necessarily concave, but they are well-known to be totally balanced, see [14], Corollary 3.2.4. Unlike concave games whose core is fully characterized and has a closed form (see Condition 1), just a single core allocation based on competitive equilibrium prices, is known for general market games, see [13], p. 266. In fact, [16] proves that a game is a market game if and only if it is totally balanced. In particular, any concave game is a market game. However, if a game is not naturally formulated as a market game (see (1)), then the task of reformulating it as a market game (or showing that such a formulation does not exist), may be as intricate as proving directly that it is totally balanced (or that it is not). Thus, it seems that except for games that are either originally stated as market games, or are easily transformed to market games, this approach has its limits.

The above stated conditions for total balancedness hold for general games. Apparently, regular games are not esoteric; many well-known games are regular and therefore deepening our understanding on the total balancedness of such games is valuable and important. In this paper we provide a couple of techniques that can help in analyzing the core of some regular games.

In the next section we rigorously define regular games. In Section 3 we present a class of regular games that we call *Aggregation Games*. For aggregation games that are non-monotone, a constructive technique that generates an auxiliary monotone aggregation game whose core is a subset of the core of the original game, is presented. We show by an example that sometimes the total balancedness of the auxiliary game is easier to identify, and such an authentication proves that the original game is totally balanced. In Section 4 we present a class of *Regular Market Games*. In particular, we develop a reduction technique that transforms certain games into market games, for which a specific core cost allocation based on competitive equilibrium prices can be derived. In Section 5 we present and study two regular queueing games that deal with servers' cooperation and show that they are regular market games and therefore they are proved to be totally balanced.

2 Regular games

Consider a cooperative game $G = (N, V)$ that is defined by its set of players N and its characteristic function V that satisfies the following conditions: Each player $i \in N$ is fully characterized by the quantities of a given number $\kappa \geq 1$ of resources that he/she owns. Let index the resources that are considered by $\ell = 1, \dots, \kappa$, so that player $i \in N$ is associated with a vector $y^i \in \Re^\kappa$, called a *vector of properties*, and y^i_ℓ specifies the quantity of resource ℓ, $1 \leq \ell \leq \kappa$, owned by player $i \in N$. Moreover, the characteristic function value of coalition $S \subseteq N$, namely $V(S)$, is a function only of the $|S|$ vectors of properties that characterize the members of S and is otherwise independent of the players' identities or the number of players in N. As we are going to show, under certain conditions it is possible to generalize the definition of such games and their characteristic function to any set of players, not necessarily those who are physically involved in the particular game $G = (N, V)$. Note that this extension is both in terms of different set of players and in the number of players. I.e., the characteristic function in such games can be applicable to any collection of vectors of properties. We call such games *Regular Games*. Apparently, the class of regular games is quite large and it contains many well-known and interesting games. In this section we will formalize the notion of regular games and propose an alternative definition and framework for such games.

As said above, regular games deal with situations where each player is fully characterized by a vector of properties, which represents her own initial amount of the $\kappa \geq 1$ resources. Some of the κ resources can be shared among the members of a coalition, where the other resources are individual resources that are not sharable. The characteristic function $V(S)$ of any coalition $S \subseteq N$, is the cost induced by the members of S when the sharable resources are used by coalition S according to the rules of the game. In some games all resources are sharable, and in other games only some resources are sharable and the others are serving as characteristics (parameters) of the players. As an example of these two types of resources, consider a number of service stations that provide different kinds of services. Each service station is associated with its own capacity and its own stream of customers. As the service stations differ in their service, the customers cannot switch upon arrival from one station to another, i.e., they are bounded to get the service from the specific station they came to. However, the service stations can share their service capacities in the sake of minimizing the overall congestion of the system. In this example, each service station is assigned a vector of properties of size 2 for its service rate and its arrival rate. The capacity is a sharable property where the arrival rate is non-sharable.

We first stress the difference between non-regular and regular games by two examples. Consider first simple games mentioned in Section 1.

Example 1 A *simple game* is a coalitional game $G = (N, V)$ where $V(S)$ is either 0 or 1, for any $S \subset N$, and $V(N) = 1$. A coalition S for which $V(S) = 1$ is called a *winning coalition*. A player who belongs to all winning coalition is a *veto player*.

In general, in a simple game the players are not associated with any quantitative property and their affect on the cost of a coalition that they join is not systematic.

Example 2 A *majority game* $G = (N, V)$ where $|N| = n \geq 3$ is odd: the game is defined by the characteristic function $V(S) = 1$ if $|S| \geq n/2$, and $V(S) = 0$ otherwise. Here we could assume that each player is associated with a vector of properties of size 1, so that the single entry of the vector is 1, and the cost of a coalition depends on the sum of the vectors of properties of the players that belong to the coalition. However, the game is non-regular as the cost of a coalition is a function of the total number of players n. That means that the cost of a specific coalition may drop from 1 to 0 by adding players and increasing n.

Next we present a well-known game, the airport game, see [11] and [12], which is later shown to satisfy the conditions of a regular game:

Example 3 An airport with a single runway serves m different types of aircrafts. An aircraft of type k, $1 \leq k \leq m$, is associated with a cost $c_k \geq 0$ of building a runway to accommodate aircrafts of its type. Let N_k be the set of airport users that use an aircraft of type k landing in a day, and $N = \cup_{k=1}^{m} N_k$ is the set of all airport users. The characteristic function for any $S \subseteq N$ is given by $V(S) = \max\{c_k | S \cap N_k \neq \emptyset,\ 1 \leq k \leq m\}$ and $V(\emptyset) = 0$. This conventional presentation of the airport game reveals its limitations as it is defined just for the given set of airport users, and, moreover, for each given aircraft type k, $1 \leq k \leq m$, it is defined just for the landings in the set N_k. It is easy to see that this definition for the game can be easily generalized to any set of players each of which having its parameter c_k as its vector of properties. This will formally be done below.

In a regular game each potential player j is associated with a vector of properties y^j of size $\kappa \geq 1$, that may be required to satisfy some feasibility constraints of the form $y^j \in D$, where $D \subseteq \Re^\kappa$. Let $y^{(m)}$ denote a sequence of m vectors of properties $y^1, \ldots, y^m$ in D. The following two definitions formally define a regular game:

Definition 1 *An infinite sequence of symmetric functions $V_0, V_1, \ldots, V_m, \ldots$ is said to be Infinite Increasing Input-Size Symmetric Sequence (IIISSS) of functions for given integer $\kappa \geq 1$, and a subset D of $\Re^\kappa$, if*

- $V_0 \equiv 0$;
- *For any $m \geq 1$, $V_m : D^m \to \Re$.*
- *There exists a vector $y^0 \in D$ such that $V_1(y^0) = 0$ and for any given sequence of $m-1$ vectors of properties $y^{(m-1)} = (y^1, \ldots, y^{m-1}) \in D^{m-1}$, $V_{m-1}(y^{(m-1)}) = V_m(y^{(m-1)}, y^0)$.*

For a given IIISSS of functions $(V_m)_{m\geq 0}$, V_m receives as input m vectors of size κ, each is a member of the set D, and it returns a real value. As the functions V_m are symmetric, the order of the m input vectors has no affect on the value of the function. In other words, let $\phi(y^1, \ldots, y^m)$ be any permutation of $(y^1, \ldots, y^m) \in D^m$. The symmetric property of the function V_m implies that $V_m(y^1, \ldots, y^m) = V_m(\phi(y^1, \ldots, y^m))$. The third item of the definition guarantees that the definition of the various functions of the IIISSS of functions is consistent, i.e., it excludes the possibility that there exist two functions V_ℓ and V_k for $\ell \neq k$, $\ell, k \geq 1$, where each is defined by a different mathematical expression. This is achieved by requiring to have a *null vector of properties* $y^0 \in D$ that connects the different functions through a forward recursion. For example, suppose that each player i is associated with a certain real number α_i that represents its liability and that the value of a coalition is the average liability of its members. In such a case let $\kappa = 2$, player i is associated with a vector $y^i = (\alpha_i, 1)$, the null vector is $y^0 = (0,0)$ and $D = \{(0,0)\} \cup \{(x,1) : x \in \Re\}$. Given m vectors of properties $y^{(m)} \in D^m$, $y^i = (\alpha_i, \beta_i) \in y^{(m)}$, $i = 1, \ldots, m$, the value $V_m(y^{(m)}) = \sum_{i=1}^m \alpha_i / \sum_{i=1}^m \beta_i$, i.e., $V_m(y^{(m)})$ is the average of the liabilities of the non-null vectors in D. Note that the choice of y^0 as the zero-vector is natural for a null vector and it holds in many other games. But in some games y^0 is not necessarily the zero vector. Consider a similar example to the above one with a characteristic function that returns for any coalition the product of the liabilities in the coalition divided by the number of players in the coalition, i.e., $V_m(y^{(m)}) = \Pi_{i=1}^m \alpha_i / \sum_{i=1}^m \beta_i$. In such a case the null vector $y^0 = (1,0)$, and $V_1(y^0)$ is defined as 0.

Definition 2 *A game $G = (N, V)$ is called regular if there exists a set $D \in \Re^\kappa$, such that player i, $i \in N$, is associated with a vector of properties $y^i \in D$, and there exists an IIISSS of functions $V_\ell : D^\ell \to \Re$, $\ell \geq 0$, such that for any $S \subseteq N$, $V(S) = V_{|S|}(y^i|_{i \in S})$.*

Observation 1 *A market game $G = (N, V)$, as defined in Condition 2 in Section 1, is not necessarily a regular game, as either the cost function of a player depends on non-quantitative properties, or alternatively, the game cannot be presented by an IIISSS of functions. Section 4 considers a class of regular games that are also market games, to be called regular market games.*

In the next two sections we identify two types of IIISSS of functions that generate many well-known regular games.

Condition 3 for total balancedness, mentioned in the Introduction, is based on the following theorem, which is proved in [2]:

Theorem 1 *A regular game based on IIISSS of functions which are both sub-additive and homogeneous of degree one*[1] *is totally balanced.*

[1]An IIISSS of functions $V_0, V_1, V_2, \ldots$ is said to be *sub-additive* if for any two finite (not necessarily disjoint) sequences of vectors of properties in D, $(y_A^i)|_{i \in A}$ and $(y_B^i)|_{i \in B}$, $V_{(|A|+|B|)}((y_A^i)|_{i \in A}, (y_B^i)|_{i \in B}) \leq V_{|A|}((y_A^i)|_{i \in A}) + V_{|B|}((y_B^i)|_{i \in B})$. It is said to be *homogeneous of degree one* if for any $m \geq 1$, $V_{m|A|}((y_A^{i(j)})_{i \in A,\ j=1,\ldots,m}) = V_{|A|}((y_A^i)_{i \in A})$, where $y_A^{i(j)}$ for $j = 1, \ldots, m$ and $i \in A$ are m replicas of y_A^i.

3 Aggregation games

In an *aggregation game*, all resources are sharable, i.e., individual properties do not exist. Moreover, when two players, say i and j, cooperate, their combine effect on the cost of the coalition they join, is as if they were replaced by a single new player having a vector of properties $g(y^i, y^j)$, for some symmetric function $g : D^2 \to D$ that satisfies the commutative and the associative laws. That means that $g(y^i, y^j) = g(y^j, y^i)$, for $i \neq j$, and that $g(g(y^i, y^j), y^k) = g(y^i, g(y^j, y^k))$ for $k \notin \{i, j\}$. Such a function g is called an *aggregation function.* For simplicity denote the aggregation function of m vectors of properties by $g^{m-1} : D^m \to D$, i.e., $g^{m-1}(y^1, \ldots, y^m) \in D$. A given cost function for one player, namely V_1, together with an aggregation function $g : D^2 \to D$, generate a corresponding IIISSS of functions in the following way: $V_m(y^1, \ldots, y^m) = V_1(g^{m-1}(y^1, \ldots, y^m))$. Next we present two example for aggregation games:

Contd. of Example 3: In the airport game $\kappa = 1$, $D = \Re_0^+$, $g : D^2 \to \Re$, where the aggregation function $g(c_1, c_2) = \max(c_1, c_2)$ for any $(c_1, c_2) \in D^2$, and $g^{m-1}(c_1, \ldots, c_m) = \max(c_1, \ldots, c_m)$ for any $(c_1, \ldots, c_m) \in D^m$. For any $c \geq 0$, $V_1(c) = c$. Thus, $V_m(c_1, \ldots, c_m) = V_1(\max(c_1, \ldots, c_m)) = \max(c_1, \ldots, c_m)$. This game is totally balanced, as for any instance of the game $G = (N, V)$ with n players that are indexed, without loss of generality, in a non-decreasing order of c_i for $i = 1, \ldots, n$, namely, $c_1 \leq c_2 \leq \ldots \leq c_n$, the cost allocation $x_i = 0$ for $i = 1, \ldots, n-1$ and $x_n = c_n$, is in the core.

Example 4 In [1] we dealt with what seems to be the simplest (but most revealing) possible model of cooperation in service systems. This model is based on the assumption that when a set of servers cooperate, they work as a single server whose service rate is the sum of the individual service rates. Moreover, this combined server serves their joint stream of arrivals. More precisely, let $N = \{1, \ldots, n\}$ be a set of n $M/M/1$ queueing systems. They can cooperate in order to minimize the steady-state congestion in the combined system. Queueing system i is associated with its own exponential service rate μ_i and its own Poisson arrival rate of customers λ_i, $\lambda_i < \mu_i$, $i \in N$. Cooperation of a set $S \subseteq N$ in this model results in a single $M/M/1$ queue whose capacity is $\mu(S) = \sum_{i \in S} \mu_i$, and whose arrival rate $\lambda(S) = \sum_{i \in S} \lambda_i$. For any coalition $S \subseteq N$ the congestion of S is given by

$$V(S) = \frac{\lambda(S)}{\mu(S) - \lambda(S)}.$$

Next we present this game as a regular aggregation game: each player, namely each queueing system, is associated with a vector of properties of size $\kappa = 2$, $y^0 = (0,0)$, and $D = \{0,0\} \cup \{(\lambda,\mu) | 0 \leq \lambda < \mu\} \subset (\Re^+)^2$. Let $V_1(y^0) = 0$, and for $(\lambda,\mu) \in D \backslash \{0,0)\}$, $V_1(\lambda,\mu) = \frac{\lambda}{\mu-\lambda}$. The aggregation function g that combines two vectors of properties in D into one is $g((\lambda_1,\mu_1),(\lambda_2,\mu_2)) = (\lambda_1+\lambda_2, \mu_1+\mu_2)$, thus $g^{m-1}((\lambda_1,\mu_1),\ldots,(\lambda_m,\mu_m)) = g(\sum_{i=1}^m \lambda_i, \sum_{i=1}^m \mu_i)$. Let $V_m(y^1,\ldots,y^m) = V_1(g^{m-1}((\lambda_1,\mu_1),\ldots,(\lambda_m,\mu_m)))$. Thus, if $g^{m-1}((\lambda_1,\mu_1),\ldots,(\lambda_m,\mu_m)) \neq y^0$, then $V_m(y^1,\ldots,y^m) = \sum_{i=1}^m \lambda_i / \sum_{i=1}^m (\mu_i - \lambda_i)$, and otherwise $V_m(y^1,\ldots,y^m) = 0$.

It was shown in [1] that this game, which is neither concave nor it possesses the shape of a market game (see (1)), is totally balanced. In fact, the non-negative part of the core has been fully characterized in [1]. In particular, $x_i = \frac{\lambda_i}{\lambda(N)} V(N)$, $i \in N$, is a core allocation.

Aggregation games are not necessarily monotone, meaning that adding players to a coalition may affect the cost of the coalition in either way: the cost may increase, decrease or it may stay intact. In fact, the game presented in Example 4 is not monotone. In [1] we suggest a procedure that may be helpful in analyzing the total balancedness of such games. For that sake we define below an auxiliary game that may be easier to analyze:

Definition 3 *Any non-monotone regular aggregation game $G = (N,V)$ is associated with another aggregation game $\tilde{G} = (N,\tilde{V})$ called its auxiliary game where $\tilde{V}(S) = \min\{V(T) : S \subseteq T \subseteq N\}$.*

The following Theorem follows from [1]:

Theorem 2

- *The auxiliary game $\tilde{G} = (N,\tilde{V})$ of a non-monotone aggregation game $G = (N,V)$ is a monotone aggregation game, with $\tilde{V}(\emptyset) = 0$, $\tilde{V}(N) = V(N)$, and $\tilde{V}(S) \leq V(S)$.*
- *If the auxiliary game $\tilde{G} = (N,\tilde{V})$ is totally balanced, then the game $G = (N,V)$ is also totally balanced.*
- *If the auxiliary game $\tilde{G} = (N,\tilde{V})$ is concave, then the non-negative part of the core of the game $G = (N,V)$ coincides with the core of the auxiliary game.*

Proof:

- The auxiliary game $\tilde{G} = (N,\tilde{V})$ is monotone by definition, i.e., for any $S_1 \subset S_2 \subseteq N$, $\tilde{V}(S_1) \leq \tilde{V}(S_2)$. Moreover, by definition, $\tilde{V}(\emptyset) = 0$, $\tilde{V}(N) = V(N)$, and $\tilde{V}(S) \leq V(S)$.

- By considering the inequalities that define the core of G it is easy to see that any core allocation of the auxiliary game $\tilde{G}$ is a core allocation of G.

- If the auxiliary game $\tilde{G}$ is concave, then by [15] each of the $n!$ extreme points of the core is a vector of marginal contribution of the players to a specific permutation of the players. As the auxiliary game $\tilde{G}$ is monotone, all the extreme points of its core are non-negative vectors, implying that its core is non-negative. Suppose by contradiction that there exists a non-negative core allocation $(x_1, \ldots, x_n)$ of G that is not a member of the core of $\tilde{G}$. That means that there exists a coalition $S \subset N$ so that $\tilde{V}(S) < \sum_{i\in S} x_i \leq V(S)$. But, $\tilde{V}(S) < V(S)$ implies, by definition, that there exists a coalition T so that $S \subset T \subseteq N$ and $\tilde{V}(S) = V(T)$. Therefore, $V(T) < \sum_{i\in S} x_i \leq V(S)$. The first inequality together with the vector $x \geq 0$ violates the fact that x is a core allocation of G, because $V(T) < \sum_{i\in S} x_i \leq \sum_{i\in T} x_i$.

■

In the problem analyzed in [1], and described in Example 4, the total balancedness of the game could be proved by showing that the auxiliary game is concave. This insight enabled us to fully characterize the non-negative part of the core.

The following example shows that an aggregation game is not necessarily balanced:

Example 5 Consider a cooperative game $G = (N, V)$, $N = \{1, 2, 3\}$, where each player i is associated with a vector of properties of size 1, namely $y^i = (\alpha_i)$, where $\alpha_i \in \{0, 1\}$. For any coalition $S \subset N$ of players let $V(S) = -\max_{i\in S} \alpha_i$. Let $y^0 = (0)$. This game is an aggregation game with $g(y^i, y^j) = \max\{y^i, y^j\}$, $V_1(y^0) = 0$ and otherwise $V_1(y^i) = -\alpha_i$. Consider $N = \{1, 2, 3\}$ with $\alpha_1 = \alpha_2 = 1$, and $\alpha_3 = 0$, thus $V(\{1\}) = V(\{2\} = V(\{1, 2\}) = V(\{1, 3\}) = V(\{2, 3\}) = V(N) = -1$, and $V(\{3\}) = 0$. The core of this game is empty. In fact, this game is a variant of a simple game, see Section 1, where $V(S) \in \{0, -1\}$ for any $S \subset N$ and $V(N) = 1$. In general, simple games are not regular games, but this specific one is.

4 Regular market games

As commented on market games in Observation 1, not all market games as defined in Condition 2 in Section 1, are regular games. We refer to all regular games that are proved to be also market games as *regular market games.* Clearly, such games are totally balanced as they are a subclass of market games. In this section we characterize a class of regular games whose IIISSS of functions, see Definitions 1 and 2, can easily be transformed into a market game as given by Equation 1. In addition, we present another class of regular games whose IIISSS of functions may not be identified at first glance as market games, but still they are convertible into market games.

In contrast to aggregation games, where players were amalgamated into one "big" player that used all their properties as given by their vector of properties, players in regular market games keep their individuality through individual quantitative properties that cannot be shared with their mates in a coalition. In market games, on the other hand, players keep their individuality, but sometimes through non-quantitative properties. In regular market games each player is associated with a vector of properties in D so that the first $s \geq 1$ properties are sharable among the players in an additive way, and the remaining $\kappa - s \geq 0$ properties are individual properties that cannot be shared. Let the set $SP = \{1, \ldots, s\}$ for some $s \leq \kappa$ be the set of sharable properties, and the set $IP = \{s+1, \ldots, \kappa\}$ be the set of individual properties that players do not share. The set D is a subset of $\Re^\kappa$ where the first s entries of the vectors are non-negative. In particular, the null vector $y^0 = \vec{0} \in D$ is the zero vector in $\Re^\kappa$. For any vector of properties $y \in D$, let $Y(y) = \{y' \in D | y'_\ell = y_\ell \text{ for } \ell \in IP\}$. The set $Y(y)$ contains all vectors of properties in D that their individual properties, i.e., the entries in IP, coincide with those of y. Let the set $E = \{y \in \Re^\kappa : y_\ell \geq 0,\ \ell \in SP, \text{ and } y_\ell = 0 \text{ for } \ell \in IP\}$. In addition, let $f : D \to \Re$, and $h : E \to \Re$, be two continuous functions with $f(\vec{0}) = h(\vec{0}) = 0$ such that for any $y \in D$

$$V_1(y) = \min\{f(x) + h(y-x) | x \leq y, \text{ and } x \in Y(y)\},\ y \in D \tag{2}$$

Equation (2) implies that $y - x \in E$. Equation (2) is well defined as the minimization is over a closed set where the first s entries of the vector x are non-negative and they are bounded from above by the respective entries of the vector y. We call the function h the *extra function.* The extra function can be null. In such a case, $V_1(y) = f(y)$ for any $y \in D$.

Theorem 3 *The class of regular market games contains all regular games that are defined by the following IIISSS of functions where $V_0 \equiv 0$, V_1 is given by Equation (2), where the function f is convex in the shared properties, the extra function h is linear in the shared properties, $y^0 = \vec{0}, y^1, \ldots, y^m \in D$,* *and for $m \geq 2$*

$$V_m(y^1, \dots, y^m) = \min\{\sum_{i=1}^{m} V_1(\tilde{y}^i) :$$

$$\tilde{y}^i \in Y(y^i) \textit{ for } i = 1, \dots, m, \textit{ and } \sum_{i=1}^{m} \tilde{y}^i = \sum_{i=1}^{m} y^i\} \tag{3}$$

Proof: It is easy to verify that Equation (3) generates regular games, see Definition 1, as symmetry of the functions V_m, $m \geq 2$, is direct. It remains to show that this game boils down to a market game. This is alos straightforward using the conditions of the Theorem together with Equation (3): Rewrite the cost function of player i, namely the function V_1, as a parametric function of its shared properties, where the parameters are her individual properties. The game then looks the same as in Condition 2 of Section 1.

∎

The following corollary is now immediate.

Corollary 1 *Regular market games are transformable to market games, thus they are totally balanced. Moreover, a core allocation for the game that is based on competitive equilibrium prices of the transformed game, exists.*

Note that the total balancedness of the games described in Theorem 3 can be argued alternatively by showing that they meet Condition 3 above. The added value of Theorem 3 and Corollary 1 is the fact that it points out a specific core allocation which Condition 3 does not. We exemplify this advantage in the two line-balancing games described in Section 5.

Example 6 This example considers a loss system that consists of a number of servers, each is associated with its own stream of customers. The system has no waiting room, meaning that a customer who finds her server busy is lost for good. It is well known that the loss probability in case of an arrival rate of λ and service rate of μ is $\lambda/(\mu + \lambda)$. Suppose there exists a set N of n servers. The arrival and service rates of server i are λ_i and μ_i, respectively, $1 \leq i \leq n$. Servers can cooperate so as to minimize their total loss rate among themselves. Assume that the servers in a coalition cannot redirect customers but they can reallocate their total service capacity so as to minimize the loss rate. Suppose that there is a cost α_i per customer lost at server i. In addition, there is an option to rent out some of the capacity of the coalition at a revenue of r per unit rate of capacity. Thus, each player is associated with the shared property $\mu_i > 0$, and the individual properties

$\lambda_i > 0$, and $\alpha_i > 0$, implying that $\kappa = 3$, $s = 1$, and $y^0 = (0,0,0)$ is the null vector. Also, $D = \{(\mu, \lambda, \alpha) : \lambda \geq 0, \mu \geq 0\}$. For $y = (\mu_i, \lambda_i, \alpha_i) \in D$, $y \neq y^0$, let

$$V_1(y) = \min_{c_i}\{\frac{\alpha_i \lambda_i^2}{c_i + \lambda_i} - r(\mu_i - c_i) : 0 \leq c_i \leq \mu_i\}.$$

This game is a regular market game as for any vector $(c_i, \lambda_i, \alpha_i) \in D \backslash \{y^0\}$ define function $f((c_i, \lambda_i, \alpha_i)) = \frac{\alpha_i \lambda_i^2}{c_i + \lambda_i}$ and the extra function $h(x, 0, 0) = rx$. As f is convex in the shared property, and the extra function is linear in the shared property, the resulting game is reducible to a market game, and therefore the game is totally balanced, and a core cost allocation based on equilibrium competitive prices can be calculated once the game is reduced to the form of Equation (3).

Future research should aim at identifying more regular games that are reducible to regular market games, as such results helps to shade light on the total balancedness of regular games. In the next section we present two examples of cooperative games in queueing systems that that are reducible to regular market games, and based on that a core cost allocation is computed.

5 Applications in line-balancing problems

Line balancing and resource pooling in service operations are an important practice. These two concepts are widely used for achieving a competitive advantage of a firm over its rivals. Some papers have considered resource pooling in the context of cooperative games, see e.g. [1], [7], [8], [9], [17], [18] and [19]. In this section we present two line balancing models, where the basic system consists of a number of $M/M/1$ queueing systems. The first model redistributes the arrival rates while holding the original servers' capacities intact, and in the second, the capacities are redistributed, while the original arrival rates are preserved. Outsourcing/renting out is allowed. In particular we answer the question of how to allocate the total cost among the servers in order to ensure stability of the grand-coalition.

5.1 The unobservable routing game with outsourcing

Consider a system that provides one kind of service by a number of servers, where each is associated with its own capacity. Each customer is a-priori assigned to a single server. This description fits, for example, clinics that specialize in one kind of treatment as vision correction surgeries, in remote locations or in under developed countries; the servers are ophthalmologists, and the customers are their clients. The clients are usually pre-examined by one of the specialists at their village. Upon arrival, a central controller routes the clients among the different ophthalmologists. The central controller has also the option of outsourcing some clients to another facility, which provides the same type of service, in order to reduce the congestion, but this comes at a cost. The objective is minimizing the sum of the congestion cost at the facility plus the outsourcing cost.

The unobservable routing game with outsourcing is defined as follows: Each server $i \in N = \{1, 2, \ldots, n\}$, is associated with an exponentially distributed service time with mean $\mu_i^{-1} > 0$, and a stream of Poisson arrivals at a rate $\lambda_i \geq 0$. A central controller reroutes the total arrival rate $\lambda(N) = \sum_{i=1}^{n} \lambda_i$ among the servers, with the option of outsourcing some at a constant cost per unit rate outsourced. Congestion is measured by the mean steady-state number of customers in the system under the optimal split of arrivals to the servers. For simplicity, assume that the cost per unit of congestion is one, and accordingly the cost per unit rate of arrivals outsourced is $b > 0$. Define a game $G = (N, V^b)$ where each coalition of servers $S \subseteq N$ is associated with a cost $V^b(S)$ that represents the optimal cost over all possible routings of the arrival rate $\lambda(S)$ among the servers of S and the external service provider.

Lemma 1 *The unobservable routing game with outsourcing is a regular market game, thus it is totally balanced for any value of b.*

Proof: For any given $b \geq 0$, the game can be formulated as a regular market game, see Section 4, with a single shared property, which is the arrival rate, and a single individual property, which is the service rate: Let $D = \{(\lambda, \mu) : \lambda \geq 0,\ \mu > 0\} \cup \{(0, 0)\}$. Let $f((z, \mu)) = \frac{z}{\mu - z}$ if $z < \mu$, and if $z \geq \mu$, let $f((z, \mu)) = \infty$. $f((z, \mu))$ represents the congestion cost at a server with an arrival rate z and service rate μ. It is easy to see that f is convex in z. Let the extra function $h((x, 0)) = bx$ represent the outsourcing cost of x customers per unit time. Thus, $V_1((\lambda, \mu)) = \min\{f((z, \mu)) + h((\lambda - z, 0)) :\ 0 \leq z \leq \lambda\}$ and $V_n((\lambda_i, \mu_i)|_{i=1\ldots n})$ for any $n \geq 2$ is defined as in Equation (3). ∎

In order to compute the core cost allocation that is based on competitive equilibrium prices we first need to reduce the game $G = (N, V^b)$ to the form of a market game. For that sake we let the function $\phi^b(\lambda, \mu)$:

$$\phi^b(\lambda, \mu) = V_1(\lambda, \mu) = \min\{\frac{z}{\mu - z} + b(\lambda - z)|\ \ 0 \leq z \leq \lambda\}.$$

Any server with a capacity rate $\mu \leq b^{-1}$ is closed at optimality regardless of the arrival rate, as it is cheaper to outsource customers then serving them by such a slow server. For such servers let $z^b(\mu) = 0$, and $\phi^b(\lambda, \mu) = b\lambda$. Otherwise, namely if $\mu > b^{-1}$, let $z^b(\mu)$ be the maximum rate of customers that this server serves them all before starting to use the outsourcing option, i.e., $z^b(\mu)$ is the solution of the equation $\frac{d(\frac{z}{\mu-z})}{dz} = b$. In general, for any $\mu > 0$

$$z^b(\mu) = \max\{0, \mu - \sqrt{\mu/b}\}. \tag{4}$$

Note that $z^b(\mu)$ is non-decreasing in b. Also, for any $\mu > 0$,

$$\phi^b(\lambda, \mu) = \begin{cases} \frac{\lambda}{\mu-\lambda} & \text{if } \lambda \leq z^b(\mu) \\ \frac{z^b(\mu)}{\mu - z^b(\mu)} + b(\lambda - z^b(\mu)) & \text{otherwise,} \end{cases}$$

and hence

$$V^b(N) = \min\{\sum_{i=1}^{n} \phi^b(z_i, \mu_i) : \sum_{i=1}^{n} z_i = \lambda(N) \text{ and } z_i \geq 0 \text{ for } i = 1, \dots, n\}. \tag{5}$$

Let the convex functions $g_i^b(z) = \phi^b(z, \mu_i)$. The game $G = (N, V^b)$ is equivalent to the following market game:

$$V^b(N) = \min\{\sum_{i=1}^{n} g_i^b(z_i) : \sum_{i=1}^{n} z_i = \lambda(N) \text{ and } z_i \geq 0 \text{ for } i = 1, \dots, n\}. \tag{6}$$

For simplicity, index the servers in a non-increasing order of their capacities, i.e., $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$. In the solution of (6), slow servers having a capacity $\mu \leq b^{-1}$ are not open no matter what is $\lambda(N)$, as outsourcing is cheaper. If $b > 1/\mu_n$, the set of open servers depends on the total arrival rate so that it shrinks as $\lambda(N)$ gets smaller. For a given $\lambda(N)$, as b increases the set of open servers grows. In general, for given $\lambda(N)$, the set of open servers is of the form $\{1, \dots, i^*(b)\}$, where $i^*(b)$ represents the index of the slowest open server, and is non-decreasing in b. Let $z^* = (z_1^*, \dots, z_n^*)$ be an optimal solution of Equation (5). Note that $z_i^* \leq z^b(\mu_i)$ implies that z_i^* is the optimal rate of customers served by server $i \in N$, where $z_i^* > z^b(\mu_i)$ implies that the rate of customers served by server i is $z^b(\mu_i)$ and a of $z_i^* - z^b(\mu_i)$, is outsourced. Thus, if $z^* = (z_1^*, \dots, z_n^*)$ is an optimal solution to Equation (5), then the optimal routing to server $i \in N$ is $\min\{z_i^*, z^b(\mu_i)\}$. The following lemma proves that the optimal routing of customers is unique, and it gives some structural properties that the vector z^* satisfies.

Lemma 2 *The cooperative game $G = (N, V^b)$ defined in (5), has a unique optimal routing of customers to the servers. In addition, either for all $i \in N$ it holds that $z_i^* \geq z^b(\mu_i)$ or for all $i \in N$ it holds that $z_i^* < z^b(\mu_i)$ or $z_i^* = 0$.*

Proof: Note that the vector $(z_1^*, \ldots, z_n^*)$ that solves $V^b(N)$ is not necessarily unique. In spite of that we claim that the rate of customers that are routed to each server is unique. The proof is based on the form of the function ϕ^b. For a given server with capacity $\mu \geq b^{-1}$, the cost function $\phi^b(\lambda, \mu)$ consists of two parts: one is the congestion cost that is paid for customers that are served by the server, namely $\min\{z^b(\mu_i), z_i^*\}$), and the second is the outsourcing cost, namely $\max\{z_i^* - z^b(\mu_i), 0\}$. As the congestion cost part in $\phi^b(\lambda, \mu)$ that is denoted by $f(z, \mu) = \frac{z}{\mu - z}$ for $z \leq \mu$ is strictly convex in $z < \mu$, the congestion that is assigned to each server at optimality is unique.

In addition, the convexity of ϕ^b implies that at optimality, the marginal cost of increasing congestion at a server is the same at all open servers, and this marginal cost is bounded from above by b. If this marginal cost is less than b then the outsourcing option is not exercised. Therefore, it is impossible that at optimality there exists two servers $i, j \in N$ with $z^b(\mu_j) > 0$, such that $z_i^* \geq z^b(\mu_i)$ and $z_j^* < z^b(\mu_j)$, as this means that the marginal cost of serving a customer at server j is lower than b, where z_i^* is already using the outsourcing option for its last customer. Such a solution can be improved by increasing z_j^* and decreasing z_i^* by the same small quantity. ■

In the rest of this subsection we show how to compute the characteristic function $V^b(N)$. We distinguish between two cases according to if the outsourcing option is utilized/not utilized by the grand-coalition N. In particular, note that if $\lambda(N) \geq \sum_{i=1}^n \mu_i$ then the outsourcing option must be used by the grand-coalition N no matter how large b is. The values $V^b(S)$, for any $S \subseteq N$, are computed in the same way.

Claim 1 *For a fixed b, the outsourcing option is not used by the grand-coalition N in the game $G = (V^b, N)$ if and only if $\lambda(N) \leq \sum_{i=1}^n z^b(\mu_i) = \sum_{\{i \in N \text{ and } \mu_i \geq 1/b\}} (\mu_i - \sqrt{\mu_i / b})$.*

Proof: The proof follows from the definition of $z^b(\mu_i)$ which is the maximum arrival rate that server i can serve at a marginal cost that is bounded from above by the outsourcing cost rate b. ■

For fixed b and $\lambda(N)$, regardless if the outsourcing option is exercised by the grand-coalition N or not, not necessarily all servers in $\{i \in N$ and $\mu_i \geq 1/b\}$ are open. In fact, some of the slow servers may still be closed. Indeed, there exists a constant Θ such that the outsourcing option is used if and only if $b < \Theta$. For the sake of specifying Θ we consider two cases:

- Case 1: The outsourcing option is not used by N. The solution to this model is described in [3] (see also [5], p.65). This case holds if $\lambda(N) < \sum_{i=1}^{n} \mu_i$ and $b \geq \Theta$. Denote the last open server in this case by i^* :

$$i^* = \min \left\{ i \in N : \quad \mu_{i+1} \leq \frac{(\sum_{j=1}^{i} \mu_j - \lambda(N))^2}{(\sum_{j=1}^{i} \sqrt{\mu_j})^2} \right\} ,$$

the optimal congestion level is

$$V^b(N) = \frac{(\sum_{i=1}^{i^*} \sqrt{\mu_i})^2}{\sum_{i=1}^{i^*} \mu_i - \lambda(N)} - i^*$$

and the now unique optimal routing rate to any open server i, $1 \leq i \leq i^*$, is

$$z_i^* = \mu_i - (\sum_{j=1}^{i^*} \mu_j - \lambda(N)) \frac{\sqrt{\mu_i}}{\sum_{j=1}^{i^*} \sqrt{\mu_j}}. \tag{7}$$

Note that as $0 \leq z_i < \mu_i$, $\frac{\partial f(z_i, \mu_i)}{\partial z_i} = \frac{\mu_i}{(\mu_i - z_i)^2}$. As $z_i^* > 0$ for $i \leq i^*$, the KKT conditions imply that $\frac{\mu_i}{(\mu_i - z_i^*)^2} = \Theta \leq b$, where Θ is the Lagrance multiplier of the constraint $\Sigma_{i=1}^{n} z_i = \lambda(N)$. Also, as this derivative at zero equals μ_i^{-1}, for any closed server $i \in \{i^* + 1, \ldots, n\}$, $1/\mu_i \geq \Theta$. Using Equation (7) for $i \leq i^*$, results in

$$\Theta = \frac{(\sum_{k=1}^{i^*} \sqrt{\mu_k})^2}{(\sum_{k=1}^{i^*} \mu_k - \lambda(N))^2}. \tag{8}$$

Note that Θ is associated with the grand-coalition. In order to stress the dependence on the coalition, let $\Theta(S)$ for $S \subseteq N$ be defined in a similar way, so that $\Theta = \Theta(N)$. The extreme case where no sub-coalition of N uses the outsourcing option occurs if only if for any $S \subseteq N$, $\lambda(S) < \sum_{i \in S} \mu_i$, and $b \geq \max_{S \subseteq N} \Theta(S) \equiv \bar{b}$. By using the ratio form of $\Theta(\cdot)$ in Equation (8), it is possible to show that $\bar{b} = \max\{\frac{\mu_i}{(\mu_i - \lambda_i)^2} : \ i \in N\}$. In this special case the vector z^* is unique as the function $f(z, \mu)$ is strictly convex in z, $0 \leq z < \mu$, and the core allocation based on competitive equilibrium prices allocates a cost $x_i = f(z_i^*, \mu_i) - \Theta(z_i^* - \lambda_i) = \frac{z_i^*}{\mu_i - z_i^*} - \Theta(z_i^* - \lambda_i)$ to server $i \in N$. That means that any server that is not open under the grand coalition, i.e., $i \in \{i^* + 1, \ldots, n\}$ pays $\Theta\lambda_i = \alpha(N)\lambda_i$ due to the service of his/her customers by other servers. An open server still pays for the congestion that he/she faces under the optimal allocation, but he/she is either compensated by Θ for an extra unit rate of arrivals (beyond λ_i) that he/she serves, or he/she needs to pay Θ per unit rate of arrivals of his/her λ_i that is routed to other servers. Equivalently, as $z_i^* = \mu_i - \sqrt{\mu_i/\Theta}$, an open server $i \le i^*$ pays $x_i = 2\sqrt{\mu_i}\sqrt{\Theta} - \Theta(\mu_i - \lambda_i) - 1$.

Upon completion of the analysis of Case 2, we present a core cost allocation based on competitive equilibrium prices for the general case, i.e., when some coalitions outsource option and the others do not. The possibility of deriving a closed form core allocation for the general case is most pronounced since all we need to deal with is the grand-coalition.

- Case 2: The outsourcing option is used by N. In this case $\Theta = b$. Let $\mu_0 = \infty$ and
$$i^*(b) = \max\{k : k \ge 0,\ \mu_k^{-1} < b\}. \tag{9}$$
  - If $b \le \mu_1^{-1}$, $i^*(b) = 0$, meaning that all servers are closed and the total arrival rate $\lambda(N)$ is outsourced. In fact, the condition $b \le \mu_1^{-1}$ implies that it is not profitable to serve any customer in-house, i.e., $V^b(S) = b\lambda(S)$ for all $S \subseteq N$. Thus, in this case, a single core allocation $x_i = b\lambda_i$ for $i \in N$, exists.
  - If $\mu_1^{-1} < b < \Theta$, the set of open servers is $\{1, \dots, i^*(b)\}$. Thus, for all $i \in \{1, \dots, i^*(b)\}$, $\frac{\partial \phi^b(z_i,\mu_i)}{\partial z_i}|z_i^* = b$.

    The solution in this case is not unique as the functions $\phi^b(z, \mu_i)$ are linear for $z > z^b(\mu_i)$ and with the same slope b. Any solution of the form $z_i^* = z^b(\mu_i) + \delta_i$, for $\delta_i \ge 0$, $i \in N$, that satisfies $\sum_{i \in N} z_i^* = \lambda(N)$ is optimal.
    Some algebra shows that
$$V^b(N) = b\lambda(N) - \sum_{i \le i^*(b)} (1 - \sqrt{\mu_i b})^2 \tag{10}$$

    As discussed above, also in this range not necessarily all subgames of (N, V^b) have their cost function of the form of equation (10), as some coalitions may not use the outsourcing option. In spite of that we present a core allocation for the general game. Note that the second term in the righthand side of (10) is the gain due to the option of in-house servicing of some of the customers.

The core allocation for the general case of $G = (N, V^b)$, which is based on competitive equilibrium prices uses Equation (6) that presents the game as a market game:

$$x_i = g_i(z_i^*) - \Theta(z_i^* - \lambda_i) \quad \text{for } i \in N \tag{11}$$

where Θ is specified in Equation (8) if the grand-coalition is not using the outsourcing option, and $\Theta = b$ if it does. As we see, Equation (11) provides a cost allocation that is in the core, where only the value for the grand-coalition, i.e., $V^b(N)$, needs to be solved.

For completeness we show that the game is not concave, ruling out the possibility of using Condition 1 in Section 1 for proving total balancedness.

Example 7 Let $N = \{1, 2, 3\}$, with $\mu_1 = \mu_2 = 100$, $\lambda_1 = \lambda_2 = 1$, $\mu_3 = 1$, $\lambda_3 = 0.99$ and $b > \bar{b} = 10^4$ so outsourcing is not used by any coalition. Let $S = \{1, 3\}$ and $T = \{2, 3\}$. We have here $V^b(\{1\}) = V^b(\{2\}) = 0.01$, $V^b(\{3\}) = 99$. In coalition S server 1 is open and likewise server 2 is open in coalition T. Thus, $V^b(S) = V^b(T) = 0.02$. In coalition $S \cup T$ servers 1 and 2 are open, each getting half of the total traffic. Hence, $V^b(S \cup T) = 2(\frac{1+0.495}{100-1-0.495})$. It is easy to see $V^b(S \cap T) = V^b(\{3\}) = 99$. Hence, $V^b(S \cup T) + V^b(S \cap T) > V^b(S) + V^b(T)$, proving that $V^b(\cdot)$ is not concave.

The next example shows a case where at least two servers are paid by the others under any core allocation, and hence no core allocation, which is either non-negative, or consisting of a single negative entry, exists. This is in contrast with our former paper [1], dealing with the model presented in Example 4, where we showed that under the type of cooperation defined there, the non-negative part of the core is non-empty, and if there exist core cost allocations with negative entries, then there exist also core allocations with a single negative entry. A negative entry means that a server is being paid in order to join this coalition. In other words, he/she is more than compensated for the waiting costs of servicing his/her own customers.

Example 8 Let $b > \bar{b}$, $n = 10$, $(\mu_1, \ldots, \mu_{10}) = (100, 78, 70, 65, 50, 45, 30, 20, 10, 5)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{10}) = (80, 60, 45, 20, 10, 20, 8, 12, 1, 4)$. $V(N) = 9.57$, which is attained when the first 8 servers are open. Let the vector $(x_1, \ldots, x_{10})$ denote a core cost allocation. Let $\ell_i = V(N) - V(N \backslash \{i\})$ and note that for any game $x_i \geq \ell_i$, $i \in N$. In particular, $x_1 > \ell_1 = 3.09$. Addionally, by considering coalition $\{1, 4\}$, we get $x_1 + x_4 \leq 3.02$ and hence $x_4 \leq -0.07$. Likewise, by considering coalition $\{1, 5\}$, we get $x_1 + x_5 \leq 2.86$ and by the same reasoning we conclude that $x_5 \leq -0.23$. That means that any core allocation comes with both servers 4 and 5 being paid by the others.

5.2 A capacity sharing model

Consider a garage with a set of n service stations $N = \{1, \ldots, n\}$. Each station provides maintenance to a particular brand of cars. It is modeled as an $M/M/1$ system, where station i is responsible for servicing a Poisson arrival rate of λ_i cars. Initially, station i is staffed by a team of workers so that its service time is exponentially distributed with parameter $\mu_i > \lambda_i$. The management is considering restaffing by reshuffling the existing manpower of capacity $\mu(N) = \sum_{i \in N} \mu_i$, among the stations in order to minimize cost. The management also considers possible reduction of the manpower. The cost of a certain configuration of capacities is given by the total congestion cost minus the savings due to the capacity reduction. For simplicity assume that the unit cost of congestion is normalized to 1, and the savings per unit reduction in the capacity rate is $b \geq 0$. The respective game $G = (N, V^b)$ is formulated below as a regular market game, where each player $i \in N$ is associated with a single shared property, which is its surplus capacity $\mu_i - \lambda_i$, and a single individual property λ_i. Let $f(z_i, \lambda_i) = \frac{\lambda_i}{z_i}$ be the congestion cost at server $i \in N$ due to a surplus capacity of $z_i > 0$. Let

$$\phi^b(z, \lambda) = \min\{f(x, \lambda) - b(z - x) |\ 0 < x \leq z\} \tag{12}$$

denote the optimal cost of a server whose arrival rate is λ and its initial surplus capacity is z. Clearly, $\frac{\partial f(z,\lambda)}{\partial z} = -\frac{\lambda}{z^2}$, $z > 0$. Let $z^b(\lambda)$ be the value of z for which this derivative equals $-b$. Thus, $z^b(\lambda) = \sqrt{\frac{\lambda}{b}}$, and

$$\phi^b(z, \lambda) = \begin{cases} \frac{\lambda}{z} & \text{if } z \leq z^b(\lambda) \\ \frac{\lambda}{z^b(\lambda)} - b(z - z^b(\lambda)) & \text{otherwise.} \end{cases}$$

The cost of the grand-coalition is

$$V^b(N) = \min\{\sum_{i=1}^{n} \phi^b(z_i, \lambda_i) : \sum_{i=1}^{n} z_i = \mu(N) - \lambda(N) \text{ and } z_i \geq 0 \text{ for } 1 \leq i \leq n\}. \tag{13}$$

The values $V^b(S)$ for any $S \subset N$ are defined in the same way. The following Lemma is similar to Lemmas 1 and 2.

Lemma 3 *The capacity sharing game $G = (N, V^b)$ defined in (13) and (12) is a regular market game, and therefore it is totally balanced. In addition, there exists a unique optimal allocation of surplus capacities to the servers.*

Proof: We prove only the last part as the rest follows directly from the proof of Lemma 1 and the convexity of the functions $\phi^b(z_i, \lambda_i)$ in z_i. Similarly to the proof of Lemma 2 for the unobservable routing game with outsourcing, the vector $(z_1, \ldots, z_n)$ that solves (13) is not necessarily unique. However, the surplus capacities allocated to the servers, namely $(x_1, \ldots, x_n)$ (see Equation (12)) are unique in view of the fact that the function $f(z, \lambda)$ used in (12) is strictly convex. Thus, there exists a unique optimal reallocation of the total excess capacity with a possible reduction of the manpower.

■

In order to derive a core cost allocation for the capacity sharing game we distinguish between two cases: The first case, where the surplus capacity is not reduced, happens when b is small enough. Let Θ be a constant such that the surplus capacity for the grand-coalition N is not reduced if and only if $b \leq \Theta$. Later we will specify the value of Θ.

- Case 1: the surplus capacity is not reduced. The total surplus capacity of $\mu(N) - \lambda(N)$ is distributed among the servers. The optimal allocation of the surplus capacity and the optimal cost are described in [10]: $z_i^* = (\mu(N) - \lambda(N)) \frac{\sqrt{\lambda_i}}{\sum_{j \in N} \sqrt{\lambda_j}}$, for $i \in N$, and

  $$V^b(N) = \frac{(\sum_{i \in N} \sqrt{\lambda_i})^2}{\mu(N) - \lambda(N)}.$$

  Note that $\frac{\partial \phi^b(z_i, \lambda_i)}{\partial z_i} = \frac{\partial f(z_i, \lambda_i)}{\partial z_i} = -\frac{\lambda_i}{z_i^2}$, for $i \in N$. At optimality, there exists a Lagrange multiplier, Θ, such that $-\frac{\lambda_i}{z_i^{*2}} = \Theta \leq -b$,$i \in N$, as the surplus capacity is not reduced. Substituting into the constraint $\Sigma_{i \in N} z_i = \mu(S) - \lambda(S)$, the z_i^* results in

  $$\Theta = -\frac{(\sum_{i \in N} \sqrt{\lambda_i})^2}{(\mu(N) - \lambda(N))^2}. \tag{14}$$

  The value of Θ is associated with N. Let $\Theta(S)$ be the respective value for any $S \subseteq N$.

  There exists $\underline{b}$ such that for $b \leq \underline{b}$, the surplus capacity of any coalition $S \subseteq N$ is fully used internally. It is easy to show that $\underline{b} = \min_{S \subseteq N} \Theta(S) = \min_{i \in N} \frac{\lambda_i}{(\mu_i - \lambda_i)^2}$. In this range the game (N, V^b) and all its sub-games do not use the capacity reduction option. The core cost allocation based on competitive equilibrium prices in this case is $x_i = f(z_i^*, \lambda_i) - \Theta(z_i^* - (\mu_i - \lambda_i))$, $1 \leq i \leq n$. Some algebra leads to

  $$x_i = 2\frac{\sqrt{\lambda_i}}{\sum_{j \in N} \sqrt{\lambda_j}} V^b(N) - \frac{\mu_i - \lambda_i}{\sum_{j \in N}(\mu_j - \lambda_j)} V^b(N), \; 1 \leq i \leq n.$$

This cost allocation is shown in [18] to be in the core of $G = (N, V^b)$ in a different way.

- Case 2: the surplus capacity is reduced. The marginal analysis here implies that $\Theta = -b$. This means that for any server $i \in N$ the optimal surplus capacity is $z^b(\lambda_i) = \sqrt{\frac{\lambda_i}{b}}$. The equalities $\frac{\partial \phi(z_i,\lambda_i)}{\partial z_i}|_{z_i^*} = -b$ for $i \in N$, imply that $z_i^* = z^b(\lambda_i) + \delta_i = \sqrt{\lambda_i/b} + \delta_i$, where $\delta_i \geq 0$, and the total reduction of capacity is $\sum_{i\in N} \delta_i = \mu(N) - \lambda(N) - \sum_{i\in N} \sqrt{\lambda_i/b}$. Substituting into (13) gives $\phi^b(z_i^*, \lambda_i) = \sqrt{\lambda_i b} - b\delta_i$, implying that

$$V^b(N) = 2\sqrt{b} \sum_{i\in N} \sqrt{\lambda_i} - b(\mu(N) - \lambda(N)) \ .$$

The core allocation for the general case of $G = (N, V^b)$ that is based on competitive equilibrium prices, uses Equation (13):

$$x_i = \phi^b(z_i^*, \lambda_i) - \Theta(z_i^* - (\mu_i - \lambda_i)) \ \text{ for } i \in N \tag{15}$$

where Θ is given by Equation (14) if the grand-coalition is not reducing its excess capacity, and $\Theta = -b$ in case it does. As we see, Equation (15) provides a cost allocation that is in the core, where only the optimization problem for the grand-coalition needs to be solved.

The next example shows that the game $G = (N, V^b)$ is non-concave.

Example 9 Using the same queueing system as stated in Example 7 with $b < \underline{b} = 1/99^2$, leading to never opting to save by reducing the surplus capacity under any coalition, results in a game which is not concave: The value of $V^b(\{1\})$ is large in comparison with the value of any other coalition so concavity is ruled out. Specifically, $V^b(S) = V^b(T) = 0.04$ where $V^b(S \cap T) = V^b(\{3\}) = 99$ and $V^b(S \cup T) > 0$. Hence, $V^b(S \cup T) + V^b(S \cap T) > V^b(S) + V^b(T)$, showing that V^b is a non-concave set function.

6 Conclusion

The main contribution of this paper is in addressing and formalizing the notion of regular games and their use. This class of games is quite large and includes many well-known games in operations management and queueing systems. We present two classes of such games: aggregation games and regular market games. We describe a couple of techniques that in some cases may help in proving that a given game is totally balanced, either by defining an auxiliary convex game whose core is contained in the core of the original game, or by reducing it to a market game. The latter case leads to the identification of a specific core allocation, the one which is based on the equilibrium competitive prices of the game it is reduced to. This is a feature which does not hold if one establishes total balancedness through the approach suggested in [2], namely identifying that the regular game is subadditive and homogeneous of degree one. Further investigation of regular games may yield new general interesting results on cooperative games.

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