A MODEL OF IMPLIED EXPECTED BOND RETURNS

by

Z. Afik*
S. Benninga**

Working Paper No 10/2013 November 2013

Research No. 01730100

* The Guilford Glazer Faculty of Business and Management Ben-Gurion University of the Negev, Israel. Email: afikzv@som.bgu.ac.il

** The Leon Recanati Graduate School of Business Administration, Tel Aviv University, Ramat Aviv, Tel Aviv, 69978, Israel. Email: benninga@tau.ac.il

This paper was partially financed by the Henry Crown Institute of Business Research in Israel.

A Model of Implied Expected Bond Returns*

Zvika Afik
The Guilford Glazer Faculty of Business and Management
Ben-Gurion University of the Negev, Israel
afikzv@som.bgu.ac.il

Simon Benninga
Faculty of Management
Tel Aviv University, Israel
benninga@tau.ac.il

This version: April 27, 2013

Abstract

Expected bond returns (EBR) are the ex-ante expectations implied by the market prices and the data set available when bond prices are quoted. Our discrete-time model can be used to estimate the rating-adjusted EBR and its risk premium components, including a certainty equivalence premium which is related to the systematic risk aversion. We apply the model to U.S. corporate bond transaction data, using rating agency transition matrices and industry specific recovery rates. We demonstrate that our model credit risk premium (CRP) is a "cleaner" measure of credit risk compared to the commonly used bond-spread. Whereas CRP versus duration term structure shows clear separation between rating groups, their parallel bond spread term structures are highly mixed raising doubts on their informational value.

JEL Classification: G12, G24, G32, G33

Keywords: expected returns, corporate bonds, credit risk

* This paper is based on Zvika Afik's dissertation at Tel Aviv University. We thank Edward Altman, Koresh Galil, Gady Jacoby, Fan Yu, and an anonymous reviewer for helpful comments. Responsibility for the paper is of course ours.

A Model of Implied Expected Bond Returns

1. Introduction

In this paper we propose a model to estimate ex-ante expected bond returns (EBR). Applying this model to market data, using rating transition matrixes and industry specific recovery rates, we compute EBR's for corporate bonds. These *expected returns* differ from the usually computed yields to maturity (*ytm*) which are the *promised returns*. The difference between the *ytm* and the EBR is a new measure of credit risk that we call the credit risk premium (CRP). We demonstrate that the commonly used bond-spread (*ytm* less the risk-free rate) is a relatively poor measure of credit-risk compared to the CRP.

We show that our reasonably simple model introduces new clarity to bond risk premia and provides evidence for the informational value of credit ratings which is often lost when bond-spreads are used. The most dramatic evidence of this can be found in Figures 9, 10, and 11 at the end of the paper, where we show that our measure of the CRP leads to separation of the yields on bonds into clearly stratified term structures.

Following the presentation of the model and its theoretical foundations, we discuss the practical estimation issues and present results using two samples of corporate bond transactions, in two periods of September-December in the years 2004 and 2011, taken from the Fixed Income Securities Database (FISD).

Whereas the stock pricing literature focuses on expected returns, the bond literature deals predominantly with yield to maturity and spreads. The yield to maturity (*ytm*) of a defaultable bond is its promised return based on promised future cash flows, if the bond is held to maturity and its issuer doesn't default. Given the positive probabilities of default on these bonds, it is clear that *ytm* is quite different from the bond's expected return.

Asset pricing theory typically focuses on *expected returns*. Good examples are the single-factor capital asset pricing model (CAPM) and the multi-factor arbitrage pricing theory (APT); in these models the expected return is derived from the appropriate risk factor loadings. Due to low liquidity of corporate bonds and data availability, empirical research of corporate bonds using such models is relatively rare. There are some exception, for examples, Gebhardt, Hvidkjaer, and Swaminathan (GHS, 2005) explore factor models for corporate bond expected returns, formulating beta sorted portfolios in the sense often found in stock returns analysis. However, typical to such models, the GHS "expected" returns are actually ex-post realized returns that are regressed on various factors and bond characteristics.[1] Our model, on the other hand, assumes that the expectations are embodied in ex-ante ("forward looking") observables such as bond ratings and market prices.

Campello, Chen and Zhang (2008) also propose an estimation methodology for expected excess bond returns, which in their paper are then used to estimate the excess equity returns. They define the expected excess return on a corporate bond as the difference between the bond yield spread and the sum of the expected default loss rate and the expected tax compensation. Their bond excess return model is based on Jarrow (1978) which assumes a diffusion process of the bond yield to maturity (a geometrical Brownian motion).

In this paper we take another approach to the estimation of expected bond returns. Our model infers the expected returns from the current market price combined with a projection of future

---

[1] For additional cross section bond realized returns analysis see also Fama and French (1993) and Elton et al (2001).

cash-flows. If one has a projection of expected cash-flows, the discount rate that matches the current market price (i.e. the IRR) is by definition the expected returns on these risky cash-flows. Obviously, the computed expected return is as good as the cash-flow projection is. Hence, in the equity market, where the future cash-flows are usually unknown, the implementation of this approach is highly questionable. In the bond market, on the other hand, we know the promised payoffs.[2] Hence *ytm*, the promised return, is very useful for bonds while there is no such parallel value in the equity market since "the promised" payoffs of a common stock are unknown. Our proposed model of EBR uses this approach. It calculates the discount rate that matches the current bond price to the expected bond payoffs. It calculates these expected payoffs using the promised payoffs and a term structure of default probability. For practical implementation we suggest using a Markov process of rating transition matrices to estimate the default probability term structure.

Although the literature on bond credit risk is vast, there are very few papers that focus on the expected returns of risky bonds modeling and estimation. A notable exception is Yu (2002), who develops a continuous-time expected returns model based on Jarrow, Lando and Yu (2005). Yu's model is relatively complex, using the one-factor CIR interest rate dynamics under the physical and the risk-neutral measure and an exponentially affine model of bond prices. Our model, on the other hand, is very simple, intuitive, easy to understand and to implement.

It is important to note here that our model does not attempt to forecast returns. We present a model of <u>implied expected returns</u> which are derived from the information set available to the modeler when a price to a bond is quoted. Whereas there is a long literature on cashflow-related expected returns of stocks,[3] we are not aware of a similar research in the corporate bond market except for Yu (2002). The risk factor approach (e.g. CAPM, Fama-French, and the APT) which is widely used in the equity market has been researched mainly in the treasury bond market, examples include Cochrane (2005 and 2008).[4]

We believe that this paper, in addition to its detailed presentation and discussion of the EBR model and the informational content of the CRP, bridges the gap between promised and expected yields of bonds.[5] Furthermore, we believe this model could be useful for research, such as Campello et al. (2008) that require expected bond returns as an input, and for practical applications of practitioners. An attractive attribute of the model is its applicability not only for data aggregation - it can also be applied to a single bond transaction as we demonstrate in the last part of this paper.[6]

---

[2] This is surely the case for straight bonds.

[3] Recent examples include Campbell and Thompson (2008), Maio (2012), and others.

[4] The few notable examples of the risk approach in the estimation of expected corporate-bond returns are listed above.

[5] An example of the inappropriateness of the *ytm* is the computation of the weighted average cost of capital $\left(WACC = \frac{E}{V} r_E + \frac{D}{V} r_D \left(1 - T\right)\right)$. Whereas the cost of equity r_E is typically computed from the security market line and hence represents the expected return to equity holders, the cost of debt r_D is usually computed as the *ytm* of the firm's debt. These two measures are incompatible. A more consistent measure of the cost of debt is the debt's expected return.

[6] Examples of individual bond transaction EBR applications include using abnormal credit risk premium for an implied bond rating application and for the estimation of implied recovery rates. We have preliminary encouraging results for both applications. These issues are beyond the scope of this paper.

The structure of the remainder of the paper is as follows: In Section 2 we present the model and develop the relations among its components. Section 3 addresses practical implementation issues, describes the data, presents and discusses a sample of estimation results. Section 4 concludes.

2. The discrete-time term-structure model of expected bond returns and yield decomposition

In this section we present the model for the estimation of expected bond returns (EBR), bond spreads and bond premia. The model also provides an intuitive economic meaning to the yield decomposition, including the credit risk premium (CRP) and the certainty equivalence premium (CEP) embedded in risky bond yields. We start with a single period model which we then extend to multiple periods. The first multiple period model applies to zero coupon bonds. We then present a coupon-bond model in which a few special cases can be solved analytically, however, the general case straight coupon-bond requires a numerical solution.

2.1 Expected bond returns: introduction and basic relations

We present below the basic definitions and relations that are used in our model. The bond yield to maturity (*ytm*) is commonly defined as the solution to equation (1):

$$(1) \qquad p = \sum_{t=1}^{T} \frac{prom(CF_t)}{(1+ytm)^t}$$

where:

$t = 1,\ldots,T$ are the payment dates

prom(CF_t) is the *promised* cash flow at date t (typically coupon payment when t<T and coupon plus principal at $t = T$)

p is the bond market price at $t = 0$.

We define the expected bond return (EBR) as the solution to equation (2):

$$(2) \qquad p = \sum_{t=1}^{T} \frac{E[CF_t]}{(1+EBR)^t}$$

where E(CF_t) is the expected cash flow of the bond at time t. The expectation is with respect to the "real" (often called "physical") probability measure and not the "risk neutral" probabilities. The EBR is thus the discount factor that prices the expected payments.

Since the default risk is the only effect included in the expected payoffs, in the nominators of equation (2) compared to equation (1), the EBR differs from the bond's *ytm* by a credit risk premium (CRP):

$$(3) \qquad ytm = EBR + CRP$$

It is easy to show that CRP $\geq$ 0 by equating the price at t = 0 in equations (1) and (2), since *prom*(CF_t) $\geq$ E(CF_t) always. CRP = 0 when the expected payoffs equal the promised payments, in a (credit) risk free bond. EBR is not the risk free rate, it is a risk-adjusted discount rate based on the bond's market price. Therefore CRP is not the commonly used bond spread; it is a new measure of credit risk.

We now calculate the expected cash flows $E(CF_t)$. Assume a discrete time processes in which the bond may be in one of two states: solvent or default. Further, assume that the bond state is observed at discrete points in time during the life of the bond, $t = 0, 1, .., T\text{-}1, T$ (see Figure 1). As time progresses from t to $t+1$ the solvent bond may default at a probability $\pi(t, t+1)$ or remain solvent at a probability 1- $\pi(t, t+1)$. Hence, at any time t, the probability of default is $\pi_{D,t}$ and the probability of the solvent state is $\pi_{S,t}$ as expressed in equations (4) and (5) respectively.

[INSERT FIGURE 1]

(4) $$\pi_{D,t} = \pi(t-1,t)\prod_{i=1}^{t-1}\left(1-\pi(i-1,i)\right)$$

(5) $$\pi_{S,t} = \prod_{i=1}^{t}\left(1-\pi(i-1,i)\right)$$

We assume that at time t a firm pays its debt holders the promised payoffs $prom(CF_t)$ in the solvent state, and in the default state the residual market value of the bond is $\delta_t \cdot principal$, where δ_t is the recovery rate of the bond at time t. Hence, the expected cash flow at time t is given by equation (6).

(6) $$E\left[CF_t\right] = \pi_{S,t} \cdot prom\left(CF_t\right) + \pi_{D,t} \cdot \delta_t \cdot principal$$

To simplify the discussion we assume in this section a frictionless market in which all securities are perfectly liquid and traded without transaction costs (and no taxes).[7] Even in such markets EBR is higher than the yield to maturity of an equivalent risk free bond with identical promised payoffs. A risk averse investor requires a premium to bear the risk of a lottery with expected payoffs $E[CF_t]$ compared to a security that pays a fixed amount of $E[CF_t]$. We call this lottery risk premium the certainty equivalence premium (CEP), which for the case of a zero-coupon bond (or for a flat term structure of r and CEP) can be expressed by:[8]

(7) $$CEP = EBR - r\,.$$

Before we turn to discuss the practical estimation of EBR of coupon bonds we analyze zero coupon bonds EBR and bond premia.

2.2 Single Period Model

To establish the basic relations and gain some essential intuitions, we start with a single-period model where we observe two traded bonds described graphically in Figures 2a and b:

R is the risk free gross return for the period $(1 + r)$

---

[7] These assumptions are used to develop the model in this section. In our data analysis we compensate for this assumption partially (by using AAA returns as a proxy to the risk-free rate).

[8] We adopted this term since adding this premium to the risky lottery return makes the investor indifferent between receiving $E[CF]$ for sure (at risk free rate) and the expected value of the stochastic, risky lottery outcome, CF (at risk free rate + CEP).

p is the price of the risky bond

π is the physical ("real") probability of default

δ is the recovery rate on the risky bond.

There are a few common definitions of recovery rate. We use a widely accepted definition: the residual value of the bond, immediately after the credit event, normalized by its face value.

[INSERT FIGURE 2a,b]

Proposition 1: In a frictionless one-period setting the following relations hold:

- The expected bond return is given by:

$$EBR = \frac{E[payoff]}{p} - 1 = \frac{1}{p} - \pi\left(\frac{1-\delta}{p}\right) - 1. \tag{8}$$

- The credit risk premium is given by:

$$CRP = ytm - EBR = \pi\left(\frac{1-\delta}{p}\right). \tag{9}$$

- The certainty equivalent premium is given by:

$$CEP = \frac{1}{p} - \pi\left(\frac{1-\delta}{p}\right) - R. \tag{10}$$

Proof: Equation (8) follows directly from the definition of EBR and the setup of Figure 2:

$$EBR + 1 = \frac{E[payoff]}{p} = \frac{1\cdot(1-\pi) + \delta\cdot\pi}{p} = \frac{1}{p} - \pi\left(\frac{1-\delta}{p}\right)$$

Equation (9) is derived by substituting the EBR of (8) into the definition of CRP of equation (3) and $1 + ytm = 1/p$, as 1 is the promised payoff of the single period. To derive equation (10), we use the definition of CEP above, it is the difference between the EBR of the risky bond and the *ytm* of a comparable credit-risk free bond in a frictionless market. In our single period setting it requires subtracting r from equation (8), resulting in:

$$CEP = EBR - r = \frac{1}{p} - \pi\left(\frac{1-\delta}{p}\right) - 1 - r,$$

which, when we write R = 1+r gives (10). ■

Discussion:

We find it useful to present the above relations of EBR, CRP, and CEP in terms of expected loss and yield reduction. In our subsequent discussion *loss* = promised payoffs less residual value, normalized by face value (which equals 1 in the above exposition).[9] Hence *loss* = 1-δ, and thus we intuitively define *yield reduction* as the *loss* divided by the bond price *p*. Equation (8) leads to a simple definition of the EBR and the CRP:

---

[9] This parallels the usual definition of loss given default (LGD) which is prevalent in the credit literature; see for example Schuermann (2004).

$$EBR = \frac{1}{p} - 1 - \pi \cdot \frac{1-\delta}{p} = ytm - \pi(yield\ \ reduction)$$

(11)
$$EBR = ytm - E[yield\ \ reduction]\ \text{, and}$$
$$CRP = E[yield\ \ reduction] = (1+ytm) \cdot E[loss]$$

where the expectations are under the physical probability measure.[10] Equation (11) gives an intuitive meaning to the credit risk premium: CRP is the expected yield reduction on the bond. It is a fraction of the gross promised yield (1+*ytm*) which is proportional to the expected loss of face value.

For further discussion of Proposition 1 and its results see Appendix C.

2.3 Multiple Period Model

We now extend the results of the previous sub-section to a multi-period framework. There is a vast volume of research and publications on tree models for bond pricing. We mention just a few of them. Black, Derman, and Toy (1990) impose a structure of a risk free interest tree based on market observed prices and volatilities. Jarrow and Turnbull (1995) focus on the default process and its integration into an interest rate (bond price) tree. Broadie and Kaya (2007) construct a binomial tree that can incorporate various "real-life" features, yet it is actually a versatile and practical implementation of structural models whereas our model is of the reduced-form type. We take a different approach: Using the process of Figure 1 we avoid the structure imposed in these richer tree models. We limits our model to two states (solvent and default); this is adequate to our model and estimation process. We present first the case of a zero coupon bond and then a few special cases of coupon bonds in a frictionless market. The general straight coupon bond requires a numerical solution and is presented in a subsequent section.

Zero coupon bond

Consider a zero coupon bond which can default at maturity, T. The risky bond pays the promised face value 1 if it does not default and its recovery value δ if it defaults. The same is relevant for the case where default might occur prior to T and the recovery is adjusted to the money market at T.[11] We maintain the prior notations and since we have a single payoff at T we suppress the time notation - the bond defaults at a probability π_D and does not default at a probability 1- π_D. This is the physical measure, we use asterisk to denote the risk-neutral measure. We also denote by R_T the (gross) risk-free interest rate for the period t=0 to t=T.

[10] Throughout this paper we use E[·] for the expectation under the physical probability measure and E*[·] under the risk neutral measure. We use similar notation for probabilities (e.g. π and π*). We extract the "real" probability of default from historical data as explained in section 3.1 "Practical Implementation" below, see equations (29), (30) and their explanation.

[11] This is known as the "Recovery of Treasury" model, where bondholders recover a fraction of the present value of face. For this definition and others, including further references, see for example Uhrig-Homburg (2002). This discussion is beyond the scope of this paper. To analyze such a case we need additional assumptions that we avoid in our present model and to account for the probability measure of default before maturity T.

Proposition 2: Consider a T-period zero-coupon bond in a friction-less market. Denote the probability of default at time T by π_D, the bond price by p, the recovery rate by δ, and the current one-period gross (i.e., one plus) interest rate by R_1.[12] Then the following relations hold:

- The expected bond return is given by:

$$EBR = \left(\frac{\left[1 - \pi_D \left(1 - \delta \right) \right]}{p} \right)^{1/T} - 1 \tag{12}$$

- The credit risk premium is given by:

$$CRP = ytm - EBR = \frac{1 - \left[1 - \pi_D \left(1 - \delta \right) \right]^{1/T}}{p^{1/T}} \tag{13}$$

- The certainty equivalent premium is given by:

$$CEP = \left(\frac{\left[1 - \pi_D \left(1 - \delta \right) \right]}{p} \right)^{1/T} - R_1 \tag{14}$$

Proof: Equation (12) follows directly from the definition of EBR and our above assumptions:

$$EBR = \left(\frac{\left[\pi_D \cdot \delta + \left(1 - \pi_D \right) \cdot 1 \right]}{p} \right)^{1/T} - 1 = \left(\frac{\left[1 - \pi_D \left(1 - \delta \right) \right]}{p} \right)^{1/T} - 1$$

To derive (13) we required an expression for *ytm* which by its definition is given by:

$$ytm = p^{-1/T} - 1 \tag{15}$$

Using (12) and (15) and the definition of CRP results in equation (13):

$$CRP = ytm - EBR = \left(\frac{1}{p} \right)^{1/T} - \left(\frac{\left[1 - \pi_D \left(1 - \delta \right) \right]}{p} \right)^{1/T} = \frac{1 - \left[1 - \pi_D \left(1 - \delta \right) \right]^{1/T}}{p^{1/T}} .$$

Equation (14) then follows directly from the relation of EBR and the risk-free rate in a frictionless world:

$$CEP = EBR - r = 1 + EBR - R_1 = \left(\frac{\left[1 - \pi_D \left(1 - \delta \right) \right]}{p} \right)^{1/T} - R_1 . \qquad \blacksquare$$

The above and subsequent relations hold for the general case, where the term structures are not flat. Each return and premium value is a function of the maturity T which is omitted in our expressions for ease of notation.

[12] In the prior section we use R for the single period gross risk free rate for single period model. Here we use R_1 in the multiperiod model to avoid confusion with R_T. Actually $R_1 = (R_T)^{1/T}$. Thus when the term structure is not flat $R_1 \neq R$ = a one period ahead rate.

For completion and for later use in the empirical section of this paper, equation (16) expresses the bond price using risk-neutral pricing.

$$p = \left[\pi_D^* \cdot \delta + \left(1 - \pi_D^*\right) \cdot 1\right] \Big/ R_T = \left[1 - \pi_D^* \left(1 - \delta\right)\right] \Big/ R_T \tag{16}$$

Hence p = PV{1 – E*[loss]} for a unit face value, where PV is the present value using the risk-free rate and E* denotes the expected value under the RN probabilities.

Consol bonds

The above analysis can be extend to the case of a consol bond with some simplifying assumptions, such as annual coupon payments and whole periods only. These assumptions can be easily relaxed for practical cases.

Proposition 3: Consider a consol bond with face value of one in a friction-less market. Assume a constant per period probability of default π_D, and a promised constant coupon rate c per period. Denote the bond price by p and the recovery rate by δ. Then the following relations hold:

- The expected bond return is given by:

$$EBR = ytm\left[1 + \pi_D\left(\frac{\delta}{c} - 1\right)\right] - \pi_D = ytm - \pi_D\left[1 - ytm\left(\frac{\delta}{c} - 1\right)\right] \tag{17}$$

- The credit risk premium is given by:

$$CRP = \pi_D\left[1 - \frac{ytm}{c}\left(\delta - c\right)\right] = \pi_D\left(1 - \frac{\delta - c}{p}\right) \tag{18}$$

Proof: We start with the well-known bond yield (*ytm*) and price relation:

$$p = \sum_{i=1}^{\infty} \frac{c}{\left(1 + ytm\right)^i} = \frac{c}{ytm} \tag{19}$$

which is consistent with the known par-valued bond relation (when p = face = 1)

We now turn to the EBR – price relation, following equation (2) and (4)-(6) and assuming a constant per period default probability π_D:[13]

$$p = \sum_{i=1}^{\infty} \frac{c \cdot \left(1 - \pi_D\right)^i + \delta \cdot \pi_D \cdot \left(1 - \pi_D\right)^{i-1}}{\left(1 + EBR\right)^i} \tag{20}$$

With simple arithmetic we have:

[13] This assumption is very common in many models using constant hazard rate. This assumption ignores rating transitions supported partially by the transition matrix whose diagonal elements are much larger than the adjacent elements.

$$p=\frac{E[CF_1]}{1-\pi_D}\cdot\sum_{i=1}^{\infty}q^i=\frac{E[CF_1]}{1-\pi_D}\cdot\frac{q}{1-q},$$

where $E[CF_1]=c\cdot(1-\pi_D)+\delta\cdot\pi_D$ is the expected cashflow at the end of period 1, and we define:

(21) $$q\equiv\frac{1-\pi_D}{1+EBR}\quad,\quad\text{thus}\quad\frac{q}{1-q}=\frac{1-\pi_D}{EBR+\pi_D}\quad,\quad\text{and}\quad p=\frac{E[CF_1]}{EBR+\pi_D}.$$

The last result is not surprising when one uses Gordon's model where period 1 payment is $E[CF_1]$ and the period growth rate is $-\pi_D$. Using some arithmetic we express EBR as a function of the other model-assumed and market-observed parameters:

(22) $$EBR=ytm\left[1+\pi_D\left(\frac{\delta}{c}-1\right)\right]-\pi_D=ytm-\pi_D\left[1-ytm\left(\frac{\delta}{c}-1\right)\right]$$

The right most expression in (22) is CRP by definition, which proves equation (18), where we use the relation p = c/*ytm* from equation (19).■

The final result on the right hand side of (18) seems very intuitive. CRP is linearly related to the per-period default probability. The multiplication factor of π_D depends on the payoff upon default. *p* is the price of the consol at any coupon date, immediately after coupon payment. The recovery less the coupon is the return on the market price *p*.[14] One minus (δ-c)/*p* is the fraction loss on the market price. This loss occurs at a probability π_D, thus CRP is a per-period expected loss (under the physical measure) intuitively.

Finite maturity straight bond

We now relax the above assumption of infinite life of a consol bond. Under the above assumptions and variable definition assume that the bond matures after T coupon periods and promises to pay its unity face value together with its last coupon, if it doesn't default at any time during its life.

Proposition 4: Consider a *T*-period coupon bond with unity face value, paid at maturity, in a friction-less market. Assume a constant per period probability of default π_D, and a promised constant coupon rate c per period. Denote the bond price by *p* and the recovery rate by *δ*. Then the following relations hold:

- The expected bond return is the solution of the following relation:

(23) $$p(T)=\frac{E[CF_1]}{EBR+\pi_D}\left(1-\frac{(1-\pi_D)^T}{(1+EBR)^T}\right)+\frac{1\cdot(1-\pi_D)^T}{(1+EBR)^T}$$

[14] Upon default the recovery "is paid" instead of the coupon, thus the difference (δ-c) is a "net" payoff upon default.

where $E[CF_1] = c \cdot (1-\pi_D) + \delta \cdot \pi_D$ is the expected cashflow at the end of period 1

- The expected bond return of a par-bond is given by:

(24) $$EBR_{@\,par} = (1-\pi_D)c - \pi_D(1-\delta)$$

- The credit risk premium of a par-bond is given by:

(25) $$CRP_{@\,par} = \pi_D(1+c-\delta)$$

Proof: Similar to equation (20) we write the price equation of a finite maturity bond:

(26) $$p(T) = \sum_{i=1}^{T} \frac{c \cdot (1-\pi_D)^i + \delta \cdot \pi_D \cdot (1-\pi_D)^{i-1}}{(1+EBR)^i} + \frac{1 \cdot (1-\pi_D)^T}{(1+EBR)^T}$$

where:

$p(T)$ is the price of a T period coupon bond (to differentiate it from p used for the consol bond above). The proof of (23) is based on equation (20) to (21).

$1 \cdot (1-\pi_D)^T$ is the face value expected payoffs at the end of period T paid if the bond had not defaulted. This characterizes a finite-maturity straight-bond,[15] the other parameters are the same as defined above for the consol bond case. For completion we repeat the known price-*ytm* relation which is actually a simple case of the above, when $\pi_D = 0$, EBR = *ytm*:

(27) $$p(T) = \frac{c}{ytm}\left(1 - \frac{1}{(1+ytm)^T}\right) + \frac{1}{(1+ytm)^T}$$

These expressions however, require numerical treatment and do not seem to lead to simple intuitive results as we have derived for the consol bond case, except for a few limiting scenarios discussed below, of a par-bond or $\pi_D = 0$.

The first limiting scenario is the special case of par-bonds. These are traded at price = 1 in our setting, such price in (27) implies the known results of c = *ytm*. Similarly, in (23) it implies:

$\frac{E[CF_1]}{EBR_{@\,par} + \pi_D} = 1$ which is identical to the price of a console bond (see (21)). Using the above par relations the proof of (24) and (25) is straightforward. ■

Equations (24) and (25) support the following conclusions:

i) The term structures of EBR and CRP are flat for par priced bonds (which holds true for *ytm* of course).

ii) At par EBR has an economic interpretation of expected payoff at the end of the first period. It equals the coupon times the probability of no default ($1-\pi_D$) plus the expected payoff upon default times the probability π_D where the recovery (δ) is paid and the par value of the bond (1) is lost.

[15] The recovery on face, "paid" upon default at a probability π_D at maturity T and the coupon c paid when default doesn't occur are included under the sigma expression of periods 1-T.

Implied recovery of finite maturity straight bond

According to equation (23) $p(\mathrm{EBR}; T, \mathrm{c}, \delta, \pi_D)$ is a well-behaved monotonically downward sloping function of EBR (when all else is fixed p(EBR) resembles the familiar shape of $p(ytm)$) and it is linear upward sloping in δ. When all parameters are fixed by market observables, data, and the assumed model, we can analyze the relations of EBR and δ:[16]

(28) $$\delta = \frac{1}{\pi_D}\left(\frac{p(T)-h}{1-h}\cdot\left(EBR+\pi_D\right)-\left(1-\pi_D\right)c\right),$$

where: $h \equiv \frac{\left(1-\pi_D\right)^T}{\left(1+EBR\right)^T}$

h has an interesting economic meaning. Its nominator is the expected payment of face value. In our case the recovery contribution upon default at time T is included in the sum expression of (26), h adds the expected contribution of the face payment at maturity, discounted appropriately by EBR. EBR is the risk adjusted discount rate which prices correctly expected payments (by definition of EBR). Therefore h is the present value of expected face payments at maturity, $\mathrm{PV}[\mathrm{E}(\mathrm{face}_T)]$. Since $\pi_D \geq 0$, and normally EBR > 0 (otherwise the pricing or the data are incorrect and are distorted economically), $0 < h < 1$. $h = 0$ only in the trivial case of $\pi_D = 1$.

When we assume an EBR for the bond, e.g. EBR^A (where the superscript A stands for assumed EBR), equation (28) expresses the recovery rate as a function of the market observables, data, the presumed model and the assumed EBR. It defines an implied recovery rate, similar to the idea of implied volatility in options models. It can be easily verified that (28) implies $\frac{\partial\delta}{\partial EBR} > 0$, thus, ceteris paribus, the recovery rate is a monotonic increasing function of EBR (and vice versa, depending on our point of view).[17] This makes sense economically. Furthermore, such one-to-one relation forms the mathematical basis for the economic idea of implied recovery. The study of implied recovery is beyond the scope of this paper.

3. Empirical results

In this section we estimate, present, and discuss empirical results of Section 2 model and relations, using FISD bond data.

3.1 Practical Implementation

We discuss below the following practical implementation matters: estimating the "real" probability of default term structure, extracting the zero-coupon term structure of interest rates

---

[16] $p(T)$ is a market observable, T and c are data, π_D is a model assumed parameter (which may practically be derived from past years default history, which can be considered a market observable since it is made available to all market participants by the rating agencies).

[17] The derivation of this result is not difficult, yet it requires some mathematical elaboration and thus omitted from this paper.

(TSIR), the assignment of recovery rate to individual firms, and selecting a proxy for the risk free rate.

Term structure of "real" default probability

We use the commonly accepted (de facto) "standard" of rating transition matrices, published by the rating agencies, to estimate the physical default probability. Our proposed estimation procedure is based on the bond rating as its risk state variable and the rating transition matrix to estimate the evolution of this state variable into "the future." We adopt this approach for its wide acceptance and usability by researchers and practitioners, yet we are aware of its limitations.

Bond ratings are under significant scrutiny by practitioners and academics. Their accuracy, consistency, and timely update are controversial and doubtful, even long before the recent market crisis.[18] Although we use the S&P ratings in this paper, our model could work with other ratings transition matrices. The literature supports the contention that there are only modest differences between the various rating systems, see for example, Schuermann and Jafry (2003).

Rating transition matrices (TM) are a key ingredient in many credit risk related models and thus are widely discussed in the literature. We mention here only a handful of sources that we find useful in our work. Schuermann (2007) provides an excellent introduction to major matters and a survey of key papers. Lando and Skødeberg (2002) emphasize the importance of continuous time estimation compared to the cohort method. Jafry and Schuermann (2004) introduce a new measure for TM comparison. Israel, Rosenthal, and Wei (2001) research the finding of generators for Markov chains via empirical TM's. The estimation accuracy of the TM has challenged researchers and practitioners, recent examples are research of confidence interval for default rates by Hanson, and Schuermann (2006), and Cantor, Hamilton, and Tennat (2007).

The model of this paper assumes a homogeneous Markov model for a bond rating and its default probability. We take the transition matrices for the Markov chain as an exogenous input from the rating agencies (S&P website for this paper). The time homogeneity is also assumed by Jarrow, Lando, and Turnbull (JLT) (1997). JLT propose a procedure to convert the physical transition probabilities to risk-neutral probabilities and use these for the valuation of risky assets. We, on the other hand, use the historical probability transition matrix to calculate the physical measure of default to estimate the yield components. When we need the risk-neutral probabilities, we estimate these from the market price of bonds without using the transition matrix. Similar to JLT, we assume that the credit migration Markov model is independent of the spot rate of interest rates.

Our assumption of Markov stationarity of the transition matrix, despite its wide popularity, is not an exact representation of the actual process. Parnes (2005) surveys the Markov rating transition literature and compares the homogeneous Markov model to several non-homogeneous alternative models. Especially intriguing are findings such as of Nickell, Perraudin and Varotto (2000) concluding: "Business cycle effects make an important difference especially for lowly graded issuers. Default probabilities in particular depend strongly on the stage of the business cycle." The "momentum effect" in bond rating transitions would challenge the assumption that transitions are Markovian (see for example Bahar and Nagpal 2001).

---

[18] Examples of bond-rating scepticism are John, Ravid, and Reisel (2005) and Löffler (2004, 2005).

The proposed implementation method

To implement our model, we generate a series of the time t state vectors s_t:

(29) $$s'_t = s'_0 \cdot \Pi^t$$

where the i^{th} component of s_t is the probability that the bond is in state (rating), $i = 1,\dots N$, and where s' is the transpose of vector s. The $\pi_{i,j}$ element of the transition matrix Π is the probability that a bond which has a rating i at the beginning of the period t would have the rating j at its end. Hence:

$$s'_t = s'_{t-1} \cdot \Pi \quad \forall t = 1,\dots T$$

assuming a time invariant Π, where T is the maturity of the bond. Since s_t is a vector of probabilities assigned to exclusive states (ratings) at time t, $\iota' \cdot s_t = 1$, where ι is the vector of ones. We define the states 1,.., N-2 to be the solvent ratings (AAA,…, C for the case of S&P ratings), the N-1 state is a default state (for a default event at time $\tau = t$) and the N^{th} state is a post default state (where default occurred in the past, at time $\tau < t$).

The above definitions and process lead to a simple calculation of the "real" probability of default (π_D) at the end of any time period t:

(30) $$\pi_{D,t} = s_{t,N-1} + s_{t,N}$$

where N-1 and N subscripts denote the last two elements of the state vector s_t.

The matrices published by the rating agencies are for a transition period of one year. However, we need transition matrices for six month period and even shorter when the first coupon payment is not whole. There are a few procedures to calculate a transition matrix for any period length. We follow Hull (2012), calculating the eigenvectors $v_1, v_2, \dots, v_n$ and the corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of the one-year transition matrix. We then use equation (31) to calculate the n^{th} root of the one-year transition matrix.

(31) $$\Pi^{1/n} = V \Lambda^{1/n} V^{-1}$$

where V and Λ are the matrices of the eigenvectors and eigenvalues respectively.

Zero coupon term structure of interest rate estimation

Fitting a "best" term structure to the noisy numerical data of interest rate has been discussed by many researchers and practitioners, see Subramanian (2001) and Hagan and West (2006) for example. We use Nelson and Siegel (NS) (1987) model for the representation and interpolation of our results. This method is often the procedure of choice by practitioners rating agencies and was found superior to alternative methods by Subramanian (2001) and others. NS parameters can be linked to common factors affecting bond returns, namely level, steepness, and curvature of Litterman and Scheinkman (1991). They found that these three factors explain on average more than 98% of the variations in bond returns. The percentages change with bond maturity and on average - level, steepness, and curvature account for 89.5%, 8.5%, and 2% of the variations respectively. Similar results are confirmed for later periods, Ramaswamy (2004) for example, shows similar results by principal components analysis on 1999-2002 data set (pp. 58-59).

NS 1987 was also scrutinized for allowing arbitrage, e.g. Bjork and Christensen (1999). Coroneo et al (2008) investigated this matter statistically and concluded that the Nelson and Siegel yield curve model is compatible with arbitrage-freeness.

For the sake of completeness we repeat NS model below:

$$r(m) = \beta_0 + (\beta_1 + \beta_2) \cdot \frac{1 - \exp(-m/\tau)}{m/\tau} - \beta_2 \cdot \exp(-m/\tau) \tag{32}$$

Where:

m is the time variable

β_0 and (β_0+β_1) are the long-term and short-term rates respectively

τ is a parameter that specifies the position of the hump

β_2 is the medium term component which determines the magnitude and the direction (up or down) of the hump or trough in the yield curve.

We need to estimate a term structure of interest rate (TSIR) for each rating category. This poses two issues:

a) An unconstrained NS curve fit often results in TSIR curve crossing, i.e. a higher rating might have a higher return than a lower rating at certain maturity ranges. This obviously has no reasonable economic support and is regarded as an undesirable artifact of the NS curve fit and data noise. A practical remedy to this issue is to use a constrained NS curve fit as follows:

   - Allow only monotonic non-decreasing term structure (*monotonicity*).
   - At any maturity the lower rating TSIR should be no lower than the next higher rating TSIR (*no-crossing*).
   - Start the curve fit with the highest rating (AAA in our case) adhering to the monotonicity requirement only. Then, one at a time, move to the nearest lower rating requiring both monotonicity and no-crossing.

b) Preferably, the curve fit is a daily one, based on a single day rich transaction data, for each day of our sample. Our data however is scattered over a period of four months. We have thus chosen to use a TSIR representative of the four month period, a single curve for each rating category. All curves are referenced to the same day. We are not aware of others that have adopted the same mechanism, yet under the assumptions of time invariant TSIR for the data period of four months we believe our methodology is theoretically sound.

For the curve fit we use the constrained non-linear optimization function of MatLab (*fmincon*), minimizing the pricing errors of all the transaction quotes of the specific rating during the data period. For consistency the market and calculated prices are discounted to the reference date of the NS curve (i.e. 1 Sep. 2004 and 2011 in our samples).

Assumed recovery rate

Upon default, our model assumes a final payment to the bond holder in the amount of a *recovery-rate* times the bond face value. We use this term to express the value of the bond at default. This definition is consistent with the study of Altman and Kishore (1996) and Moody's publications that use market value of defaulted bonds. However, recoveries may relate to the value of the bond at the end of the distressed-reorganization period (see Altman and Eberhart 1994). [19] We ignore the effect of value changes after default on a bond yield.

Our main reference for recoveries on defaulted bonds is the research of Altman and Kishore (AK, 1996) that tabulates average rates by SIC sectors. We compared AK recoveries with more recent Moody's data, the recovery rates are not identical, yet not significantly different from that of AK. We prefer AK data since it links SIC codes of industry sectors with the recovery rates, whereas other publications are less specific, utilizing only short descriptive names to identify the respective industry.

AK find no relation between the recovery rate and the original rating of a bond issue, once seniority is accounted for. In addition, they conclude that neither the size of the issue nor the time to default from its original date of issuance has any association with the recovery rate. The validity of these findings to specific market conditions remains an open issue.

Within each SIC code group, AK find quite dispersed data, with standard deviation that are mostly in the 20-28% range.[20] We attribute the industry segment median recovery rate to each bond according to Table 3 of AK. We are aware of the fact that this is not an accurate match of recovery to individual firms and bonds, yet this seems the best practical choice among the applicable alternatives.

Risk free rate

The US government bond TSIR is often used as a benchmark risk free rate. This is also our choice for the calculation of the bond-spreads. However, when we need to control for the effects of liquidity premium and taxes to improve our estimates of the risk-neutral default probability term structure we need an alternative to the treasury TSIR. It is well known, and supported by the data of this work, that the credit premium and real default probabilities are almost negligible for AAA rated bonds, yet they command a significant “other” premia above the treasury TSIR. Thus, we adopt the AAA TSIR as our risk-free benchmark, assuming it nets out (at least partially) the liquidity and tax effects of the other ratings.

We acknowledge that there are liquidity differences among the various ratings and that taxes may affect differently speculative bonds and investment grade ones, yet we ignore these secondary effects in our current research.

3.2 Sample results and discussion

The following describes and discusses our results of estimating the above yields, premia, and default probability measures aggregate term structures of zero coupon bonds and EBR and CRP of individual coupon bonds using two samples of US corporate bonds.

3.2.1 The data

A complete and reliable corporate bond data remains one of the challenges in our research. For the current research we use The Fixed Income Securities Database (FISD). It covers over

[19] Uhrig-Homburg (2002) and Bakshi , Madan, and Zhang (2004) describe common definitions of the recovery rate.

[20] In Moody's publications, for example, we did not find recovery rate dispersion within industry sectors.

100,000 corporate, U.S. Agency, U.S. Treasury and supranational debt securities and includes more than 400 fields or data items. FISD might be biased to bond portfolio activity of insurance companies including infrequent trading, biased to initial offer, large volumes, long-term holding periods, and other specific sample and price biases. We processed and analyzed the transactions of two periods, each four months long, spanning September to December of 2004 and 2011. We believe these two periods are far enough apart, one well before the 2007-8 meltdown and the other very recent, demonstrating the nature of the model in "normal" market state. The analysis of the model during the turbulent period of 2007-2009 is beyond the scope of this work and planned for a subsequent paper.

We used only corporate bonds, rated by S&P, that pay fixed coupons semiannually. All other were excluded from our data set. We also excluded other data including:

- Non-straight bonds such as callable, putable, and convertible issues.
- Bonds with less than 3 coupons remaining.
- Transaction data without a price quotation (or non positive price)
- Bonds for which we failed matching a complete set of inputs required for our calculations.

Our initial raw data includes 116,899 and 88,074 transaction lines for the 2004 and 2011 four months periods respectively. After filtering and fusion of data from other tables we have 12,936 and 6,066 transaction lines with complete data sets for our analysis (for 2004 and 2011 periods respectively). The statistics of the net final set are presented in Appendix B. It is interesting to note that most of the data in our samples is of senior (unsecured) bonds, other seniority types are of negligible amount.

For the US treasury TSIR we used daily historical interest rates (published on Yahoo Finance) matching the corporate bond sample periods for constant maturities of 0.25, 5, 10, and 30 years. We interpolated this data for intermediate maturities.

The transition matrices that we use are taken from S&P publications: Global Average One-Year Transition Rates, 1981 to 2003 and 1981 to 2010 for the 2004 and 2011 sample periods respectively.

3.2.2 Results examples and discussion

Zero-coupon bond term structure of returns

First we estimate the TSIR of the S&P rating categories for our data set. As explained above, for each rating we estimate the four NS parameters defining the rating TSIR. To avoid undesirable crossing of fitted curves we impose certain constrains in our estimation process as explained above. Others, such as Diebold and Li (2006) have chosen to estimate a linear model of NS β_0,β_1,β_2 parameters and imposed a constraint on the fourth (non-linear) parameter $\tau = 1.3684$. This value is not remote from our results, yet we do not see a reason to impose this constraint in our case.

Figures 3a and 3b show the TSIR of our NS fit for zero coupon bonds extracted from our data for investment and speculative grade respectively. Figure 3b shows that the C-CCC graded bonds actually form a separate class in our data set. These figures show that in our dataset there is a distinct clustering of the TSIR's to the three major rating categories A, B, and C, with pronounced spreads between them compared to the intra-group spreads.

[INSERT FIGURE 3a,b]

Default probabilities term structure

We compute π_D "real" probabilities as explained in section 3.1 above using S&P transition matrices. For the calculations of risk-neutral (RN) default probabilities π_D* we use equation (16). It expresses the relation among the transaction price *p*, observed in the market, the risk free rate R_T, which we assume is proxied by the AAA TSIR, the recovery rate δ, which we assume equals 40% for all ratings (just for the aggregate term structure here), and the estimated π_D*. [21]

Figures 4a,b show a sample of "real" and RN default probabilities of AA and BBB bonds for the 2004 sample. The results for other bond ratings are similar and monotone.[22]

[INSERT FIGURE 4a,b]

These results are in agreement with results obtained by Delianedis and Geseke (2003) who compute risk-neutral default probabilities using the option-pricing based models of Merton (1974) and Geske (1977). They show, not surprisingly, that their estimates for the risk-neutral default probabilities from both models exceed rating-migrations based physical default probabilities.

Bond yield decomposition

Using the above estimated zero-coupon returns and default probability term structures of each of the rating categories, we estimate the expected bond returns (EBR), the credit risk premium (CRP) and the certainty equivalence premium (CEP) term structures using equations (12), (13), and (14) respectively. Figure 5a,b shows the term-structures of EBR, CEP, and CRP of zero coupon bonds rated AA and BB respectively. Other rating group results are omitted for brevity.[23]

[INSERT FIGURE 5a,b]

It is worthwhile recalling that $(1+\text{EBR})^T = (1 - \text{E}[loss])/p = (1+ytm)^T - \text{E}[loss]/p$ where the expectations are under the "real" probability measure (see Proposition 2). CRP is the difference between *ytm* and EBR and thus expresses a per-period related expected loss rate (of the promised gross yield 1+*ytm*).[24] The CEP in a frictionless world is the difference between EBR and the risk-free rate. In our estimation we assume that the market friction premia (mainly liquidity, transaction cost, bid-ask, and tax effects) are embodied in AAA rated zero coupon bond returns. Thus we estimate CEP of rating *j* bonds by:

$$\text{CEP}_j(T) = \text{EBR}_j(T) - ytm_{\text{AAA}}(T) \quad \text{for each maturity } T.$$

[21] As a more robust alternative, which does not require an assumed recovery rate, one may estimate the risk-neutral expected loss: $E^*[\text{loss}] = \pi_D^* \cdot (1-\delta)$.

[22] Not included in this paper due to space limitation; available from the authors.

[23] We have also similar results for the period September-December 2011. These are not included in this paper due to space limitation, yet are available from Zvika Afik by request.

[24] Using an approximation based on the general Binomial Theorem, the 1/T power expressions of the multi-period zero coupon expressions of Proposition 2 can be simplified (practically a good approximation for the values encountered in real market data). This results in expressions of per-period loss such as: $\left[1-\pi_D(1-\delta)\right]^{1/T} \approx 1-\frac{1}{T}\pi_D(1-\delta)$

EBR increases monotonically with declining bond rating, and usually with maturity. CRP monotonically increases with maturities and decreasing bond rating. CEP term structure exhibits a humped shape, reaching a maximum at maturities of 10-15 years and then moderately decreases. This however might be a result of the assumed flat term structure of recovery rate. Dwyer and Korablev (2009) present a model for the estimation of such term structures based on a Markov transition process of recovery states. As a result, one may expect a decreasing (increasing) term structure of recovery rate for an investment (speculative) grade bond. Even under our simple assumption, of a flat historical average recovery rate, CEP generally increases with decreasing rating. This is expected, as the CEP expresses the risk aversion to the lottery, the additional discount investors demand of EBR above the risk-free rate.[25]

Coupon bonds EBR term structure

Figures 6a and 6b present examples of coupon bonds EBR term structure for BBB and AAA bonds respectively for the period September-December 2011. We represent the time by bond duration and not by bond maturity as the duration captures the coupon effect and thus represents the payoff center of gravity (better than the maturity). Figures 7a and 7b are similar, include the results of September-December 2004 sample for comparison.[26] As is typical of the corporate bond market *ytm* observations are quite scattered. EBR observations a similarly scattered, yet mostly they appear to be located at a characteristic distance below their respective *ytm* observations (a distance we define as the CRP). This distance generally grows for lower credit rating and longer durations as expected. In the next section we demonstrate that the CRP term structure is typical for each credit rating.

[INSERT FIGUREs 6-7]

Comparing coupon bonds' CRP to bond-spreads

To compare CRP term structure to that of bond-spreads we plotted these variables versus duration, by rating groups, see Figures 8-10 for CRP and bond spreads of AAA/AA/A, BBB/BB/B and AA/A/BBB ratings respectively for our 2011 sample and Figure 11 for the 2004 sample. These figures demonstrate clearly that bond spread is a very noisy measure of credit risk. The bond-spreads term structures of adjacent rating classes are highly mixed, showing poor monotonicity and relation with credit rating, raising doubts on the informational value of this measure (or alternatively of the ratings or both). The CRP results, on the other hand, cluster around clear term structures characteristic to each credit rating group, showing clear separation between rating classes.

[INSERT FIGUREs 8-11]

To explain the relation and significant difference between bond spreads and CRP we decompose the risky bond promised returns ytm^b as follows:

$$ytm^b = CRP + EBR = CRP + ytm^{rf} + CEP + LP + TS + OP \tag{33}$$

---

[25] It is worth recalling that CEP is analogous to the CAPM "systematic risk". This relation is presented in Appendix C and equation (43).

[26] The figure captions state the number of observations included in each chart and its respective Nelson-Siegel curve fit. These numbers reflect the deletion of a few extreme and erroneous observations to avoid curve-fit distortion and user-unfriendly chart scale.

where:

ytm^{rf} is the yield to maturity of an equivalent ideal, riskless bond, with the same coupon rate, terms, and time to maturity (it is the r^f equivalent for a coupon bond).

LP is the Liquidity Premium, which is zero for a bond that can be purchased and sold at any time and at any quantity at its fair value.

TS is the Transaction Spread (e.g. practically half the bid-ask and other fees).

OP includes all Other Premia (such as tax effects).[27]

From equation (33) it is obvious that the bond spread, includes all the premia on the right-hand-side excluding the risk-free rate whereas CRP is based on the difference between the promised and the expected yields, both include CEP, LP, TS, OP and the risk-free return. In practical applications the bond spread is often calculated using zero coupon treasury rate, which adds "noise" related to the coupon effect. This can be partially neutralized using duration for the time variable as we did in this section.

4. Conclusions

This paper presents a simple and practical model of market data implied expected bond returns, based on expected bond cash-flows. By nature this model calculates ex-ante expected returns of defaultable bonds and thus helps addressing the need for forward looking expected returns which are otherwise often estimated using realized ex-post historical returns.

Our model does not attempt to forecast returns. We present a model of implied expected returns which are defined as the IRR of the bond expected cashflows. Hence our model is as good as our estimation of these expected cashflows.

We start with an idealized approach in which we have a complete set of forward looking default probability term structure and a matching set of recovery rate term structure (Figure 1). In a perfect forward information situation, these term structures would change over time, embodying the information of the whole market, the specific issuer and the specific bond. Having such perfect forward looking information set is obviously impractical and thus we present implementable approximations which rely on fixed transition matrices and assumed recovery rates.

We develop a simple, intuitive, and practical framework for modeling and estimating the term structure of zero-coupon expected bond returns and the decomposition of bond yields.

We estimate these yields and premia using US corporate straight-coupon-bond market data of the period September-December 2004 and a parallel period in 2011. As a by-product of this process we also estimated a term structure of default probabilities for each bond rating under the risk neutral and the "physical" measures.

Since we estimate the expected bond returns (EBR), the credit risk premium (CRP), and the certainty equivalence premium (CEP) from market prices and the information set available to the investor, EBR, CRP, and CEP are implied by the market information and may change as a result of market prices for example. We do not regard such dependence on prices as a weakness. On the contrary, assuming market prices embody the collective expectations of

[27] Other Premium compensates for other variables and unknowns (added for the completeness of the model and can be regarded as the error term / innovation in an estimation model).

future payoffs, our model "calibrates" the ex-ante term structures of EBR, CRP, CEP, and risk-neutral default probabilities to these timely expectations.

After estimating these term structures, aggregating market data for each rating group in a zero coupon modeling we demonstrate the usability of the model for individual coupon bonds. Not surprisingly (see equation (33) and its discussion), the CRP generates term structures with clear delineation between rating groups, unlike the commonly used bond spreads. This characteristic can be used for models of market implied rating and market implied recovery rates which are beyond the scope of this work and are the topics of forthcoming research.

We focus this work on developing the model and the empirical estimation process. Therefore our empirical results are limited to two four-month periods in 2004 and 2011. The results are encouraging as they are consistent despite the fact that these periods are far apart chronologically and economically (given the crisis that started in mid-2007). We plan on expanding our data set and explore the model results on additional periods.

We demonstrate the estimation using unconditional rating transition matrices. A similar estimation seems feasible using conditional default probabilities that could be estimated using hazard models. We believe that these open new opportunities for researchers and practitioners.

References

Altman, E.I. and Allan C. Eberhart, 1994. "Do Seniority Provisions Protect Bondholders' Investments," *Journal of Portfolio Management*, Vol 20, no 4 (Summer): 67-65.

Altman, E., and V.M. Kishore, 1996. "Almost Everything You Wanted to Know about Recoveries on Defaulted Bonds," *Financial Analysts Journal*, (November/December): 57-64.

Bahar R. and K. Nagpal, 2001. "Dynamics of Rating Transition," *Algo Research Quarterly*, 4 (Mar/Jun), Pages 71-92.

Bakshi G., D.B. Madan, and F.X. Zhang, 2004. "Understanding the Role of Recovery in Default Risk Models: Empirical Comparisons and Implied Recovery Rates," EFA 2004 Maastricht Meetings Paper No. 3584; AFA 2004 San Diego Meetings; FEDS Working Paper No. 2001-37.

Bao J., J. Pan, and J. Wang, 2011. "The Illiquidity of Corporate Bonds," *The Journal of Finance*, 66, (3), 911-946.

Bjork, T., and B.J. Christensen, 1999. "Interest Rate Dynamics and Consistent Forward Rate Curves." *Mathematical Finance* 9, 323–348.

Black, F., E. Derman, W. Toy 1990. "A One-Factor Model of Interest Rates and Its Application to Treasury Bond Options." *Financial Analysts Journal* (Jan/Feb), 46, 1.

Broadie, M. and O. Kaya, 2007. "A Binomial Lattice Method for Pricing Corporate Debt and Modeling Chapter 11." *Journal of Financial and Quantitative Analysis* (Jun), 42, 2.

Campbell, F.Y., 2007. "Estimating the equity premium." NBER Working Paper 13423.

Campbell, F.Y., and S.B. Thompson, 2008. "Predicting Excess Stock Returns Out of Sample: Can Anything Beat the Historical Average?" *Review of Financial Studies*, 21 (4), 1509-1531.

Campello, M., L. Chen, and L. Zhang, 2008. "Expected returns, yield spreads, and asset pricing tests." *Review of Financial Studies*, 21(3), 1297-1338.

Cantor, R., D.T. Hamilton, and J. Tennat, 2007. "Confidence Intervals for Corporate Default Rates." SSRN.

Cochrane, J., and M. Piazzesi, 2005. "Bond risk premia." *American Economic Review*, 95, 138–160.

Cochrane, J., and M. Piazzesi, 2008. "Decomposing the Yield Curve." Working Paper, SSRN.

Coroneo, L., K. Nyholm, and R. Vidova-Koleva, 2008. "How Arbitrage-Free is the Nelson-Siegel Model." Working Paper No. 874 (February), European Central Bank.

Delianedis, G. and R.L. Geske, 2003. "Credit risk and risk neutral default probabilities: information about rating migrations and defaults." UCLA Working Paper.

Diebold, F. X. and C. Li, 2006. "Forecasting the Term Structure of Government Bond Yields." *Journal of Econometrics*, 130, 337-364.

Dwyer, D. and I. Korablev, 2009. "Moody's KMV LossCalc V3.0," Moody's Analytics, manuscript.

Elton, E. J., 1999. "Expected return, realized return, and asset pricing tests." *Journal of Finance*, 54: 1199-1220.

Elton, E.J., M.J. Gruber, D. Agrawal, and C. Mann, 2001. "Explaining the rate spread on corporate bonds." *Journal of Finance*, Vol. 56, No. 10, 247-277.

Fama, E. and K.R. French, 1993. "Common risk factors in the returns of stocks and bonds." *Journal of Financial Economics,* 33, 3–56.

Gebhardt W.R., S. Hvidkjaer, and B. Swaminathan, 2005. "The Cross-Section of Expected Corporate Bond Returns: Betas or Characteristics?" *Journal of Financial Economics,* 75, 85-114.

Geske, R., 1977. "The Valuation of Corporate Liabilities as Compound Options." *Journal of Financial and Quantitative Analysis*, 541-552.

Goyenko, R., A. Subrahmanyam, and A. Ukhov, 2011. "The Term Structure of Bond Market Liquidity and Its Implications for Expected Bond Returns." *Journal of Financial and Quantitative Analysis*, 46, (1), 111–139.

Hagan, P., and G. West, 2006. "Interpolation methods for yield curve construction." *Applied Mathematical Finance,* Vol. 13, No. 2, 89–129.

Hanson, S., and T. Schuermann, 2006. "Confidence intervals for probabilities of default." *Journal of Banking & Finance*, 30, 2281–2301.

Heinen, A., 2012. "A Comment on Bond risk, bond return volatility, and the term structure of interest rates," *International Journal of Forecasting*, 28, 118–120

Hull, J., 2005. Technical Note No. 11 to Options, *Futures, and Other Derivatives*. 8th ed. (published on Hull's website)

Ilmanen, A., 2003. "Expected Returns on Stocks and Bonds," The Journal of Portfolio Management, (Winter), 7-27.

Ilmanen, A., 2005. "Time-Varying Expected Returns in International Bond Markets," *The Journal of Finance*, 50, (2), 481-506.

Israel, R.B., J.S. Rosenthal, and J.Z.Wei, 2001. "Finding Generators for Markov Chains via Empirical Transition Matrices with Applications to Credit Ratings." *Mathematical Finance,* (April), 245-265.

Jafry, Y. and T. Schuermann, 2003. "Metrics for Comparing Credit Migration Matrices," Wharton Financial Institutions Working Paper #03-08.

Jafry, Y. and T. Schuermann, 2004. "Measurement, Estimation and Comparison of Credit Migration Matrices." *Journal of Banking & Finance*, 28, 2603-2639.

Jarrow, R., 1978, "The relationship between yield, risk, and return of corporate bonds, *Journal of Finance*," 33, 1235-1240.

Jarrow, R.A., D. Lando, and S.M. Turnbull, 1997. "A Markov model for the term structure of credit spreads" *Review of Financial Studies*, 10 , 481–523.

Jarrow, R.A., D. Lando, and F. Yu, 2005. "Default Risk and Diversification: Theory and Applications," *Mathematical Finance,* 1, (January), 1-26

Jarrow, R.A., and S.M. Turnbull, 1995, "Pricing derivatives on financial securities subject to credit risk," *Journal of Finance,* 50, 53-85.

John, K. S., A. Ravid, and N. Reisel, 1995. "Senior and Subordinated Issues, Debt Ratings and price Impact," working paper.

Lando, D. and T. Skødeberg, 2002. "Analyzing Ratings Transitions and Rating Drift with Continuous Observations." *Journal of Banking & Finance* 26, 423-444.

Litterman, R. and J. Scheinkman, 1991. "Common Factors Affecting Bond Returns." *Journal of Fixed Income*, (June), 54-61.

Löffler, G., 2004. "An anatomy of rating through the cycle." *Journal of Banking and Finance*, 28, 695-720.

Löffler G. 2005. "Avoiding the Rating Bounce: Why Rating Agencies Are Slow to React to New Information." *Journal of Economic Behavior & Organization*, Vol. 56, Issue 3 (Mar), pp. 365-381.

Maio, P. 2012. "The "Fed Model" and the Predictability of Stock Returns." *Review of Finance* (forthcoming).

Merton, R. 1974. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates." *Journal of Finance* 2(2): 449-470.

Nelson, C. and Siegel, A. 1987. "Parsimonious Modeling of Yield Curves." *Journal of Business* 60: 473-489.

Nickell P., Perraudin W. and Varotto S., 2000. "Stability of Rating Transitions." *Journal of Banking and Finance*, 24: 203-227.

Parnes, D. 2005. "Homogeneous Markov Chain, Stochastic Economic, and Non-Homogeneous Models for Measuring Corporate Credit Risk," working paper

Ramaswamy, S. 2004. *Managing Credit Risk in Corporate Bond Portfolios - A Practitioners Guide*, Wiley, New Jersey.

Schuermann, T. 2004. "What do we know about loss-given-default," Wharton Financial Institutions Center Working Paper #04-01.

Schuermann, T. 2007. "Credit Migration Matrices," *Risk-Encyclopedia*.

Schuermann, T. and Y. Jafry, 2003. "Measurement and Estimation of Credit Migration Matrices," Wharton Financial Institutions Center Working Paper #03-09.

Subramanian, K.V., 2001. "Term Structure Estimation in illiquid Markets," *The Journal of Fixed Income*, (June): 77-86.

Uhrig-Homburg, M., 2002, "Valuation of defaultable claims: A survey," *Schmalenbach Business Review*, 54, 24-57.

Vazza, D. and D. Aurora, 2004. "S&P Quarterly Default Update & Rating Transitions." S&P Global Fixed Income Research, (October)

Viceira, L.M., 2012. "Bond risk, bond return volatility, and the term structure of interest rates," *International Journal of Forecasting*, 28, 97–117

Yu, F., 2002. "Modeling Expected Return on Defaultable Bonds," *Journal of Fixed Income*, 12(2): 69-81

Figures

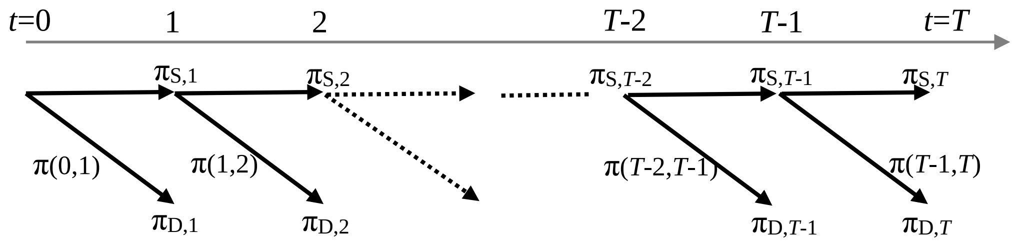


Figure 1: Bond credit state evolution. A solvent bond continues on the horizontal path from t=0 to t=T, whereas a defaulting bond drops down to a default state at $t \in \{1,..T\}$. As time progresses from t to t+1 the solvent bond may default at a probability $\pi(t, t+1)$ or remain solvent at a probability 1- $\pi(t, t+1)$. Hence, at any time t, the probability of default is $\pi_{D,t}$ and the probability of the solvent state is $\pi_{S,t}$.

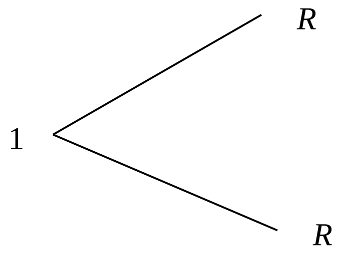


Figure 2a: A risk free bond

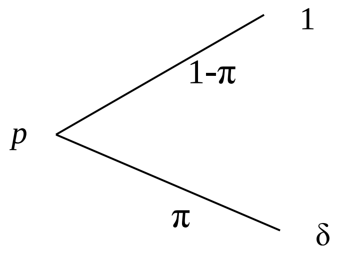


Figure 2b: A risky bond

R is the risk free rate, p is the bond price, δ the recovery rate, and π the default probability (physical measure)

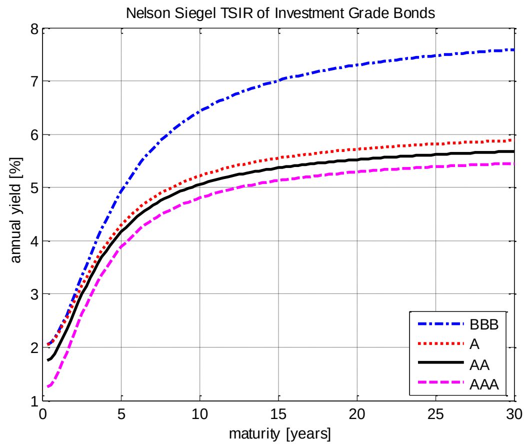


Figure 3a: investment grade bonds

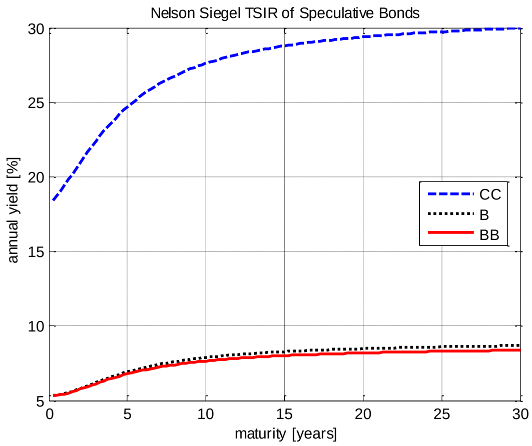


Figure 3b: speculative bonds

The term structure of interest rates for zero coupons by rating groups (Sep-Dec 2004)

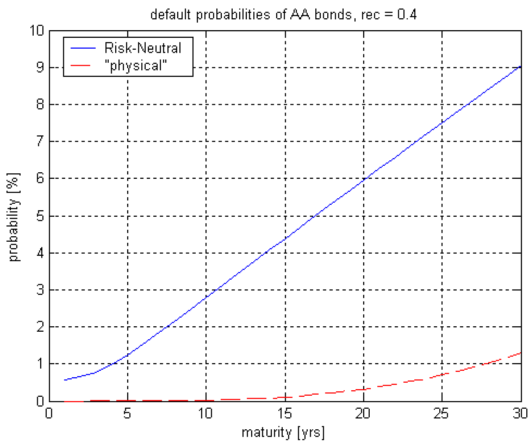


Figure 4a: AA rated bonds

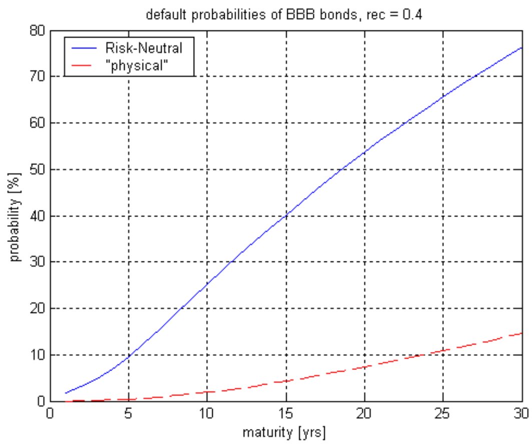


Figure. 4b: BBB rated bonds

Term structure of physical and risk-neutral probabilities of default (Sep-Dec 2004 sample)

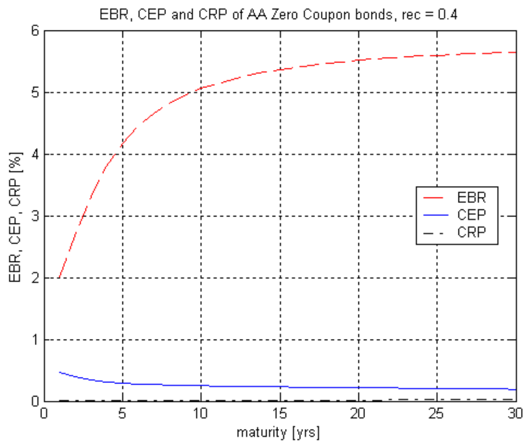


Figure 5a: AA rated bonds

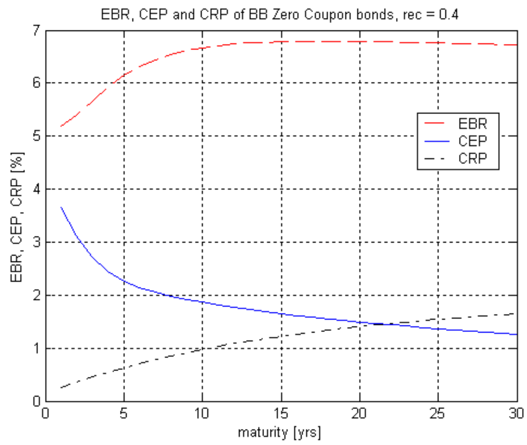


Figure 5b: BB rated bonds

Zero coupon bonds term structure of expected bond returns (EBR), certainty equivalence premium (CEP), and credit risk premium (CRP), assuming 40% recovery rate**,** Sep-Dec 2004 sample.

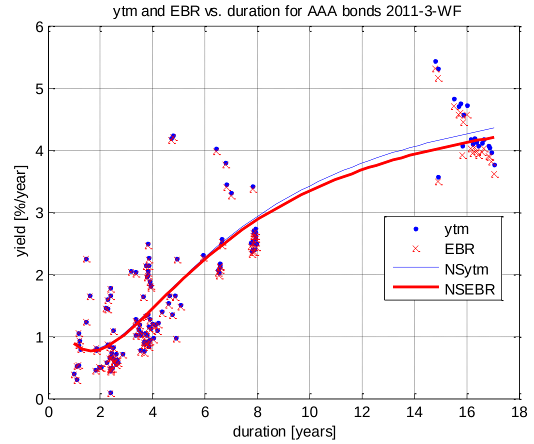


Figure 6a: AAA rated bonds (164 obs.)

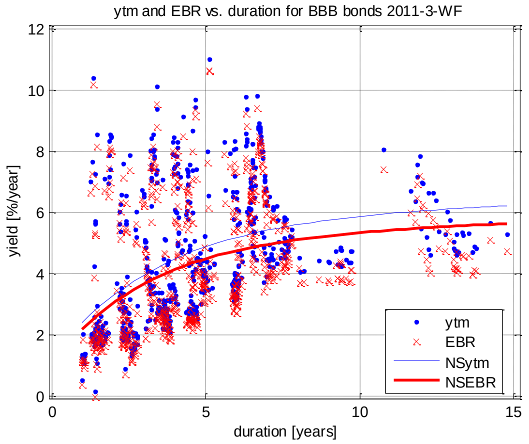


Figure 6b: BBB rated bonds (891 obs.)

Coupon bonds EBR and *ytm* versus duration (returns are in %, duration in years). For each reported transaction in the filtered sample we calculated its EBR (red x) and *ytm* (blue dot). The Nelson-Siegel curve fits are the heavy red and fine blue lines, for EBR and *ytm* term structure respectively**,** Sep-Dec 2011 sample.

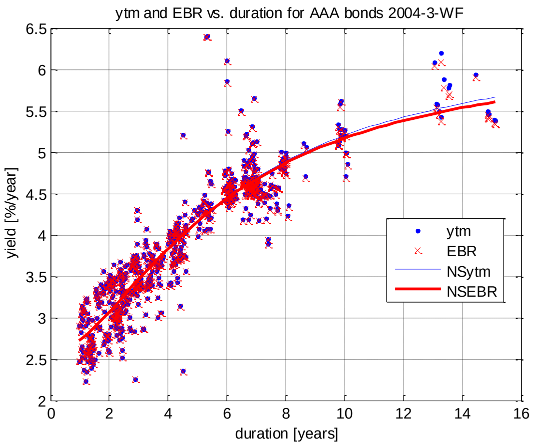


Figure 7a: AAA rated bonds (937 obs.)

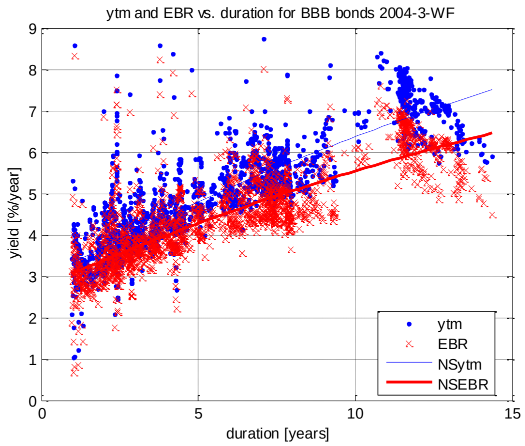


Figure 7b: BBB rated bonds (2,758 obs.)

Coupon bonds EBR and *ytm* versus duration (returns are in %, duration in years). For each reported transaction in the filtered sample we calculated its EBR (red x) and *ytm* (blue dot). The Nelson-Siegel curve fits are the heavy red and fine blue lines, for EBR and *ytm* term structure respectively, Sep-Dec 2004 sample.

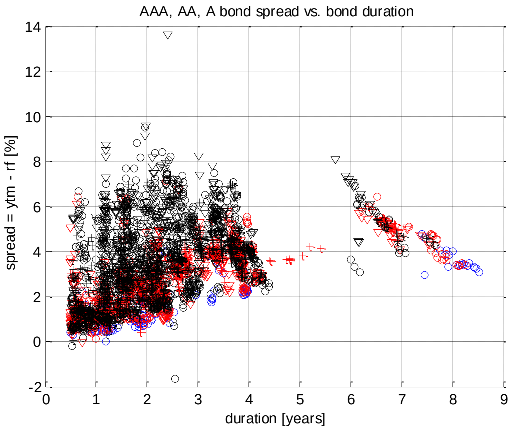


Figure 8a: "A" bond spread versus duration

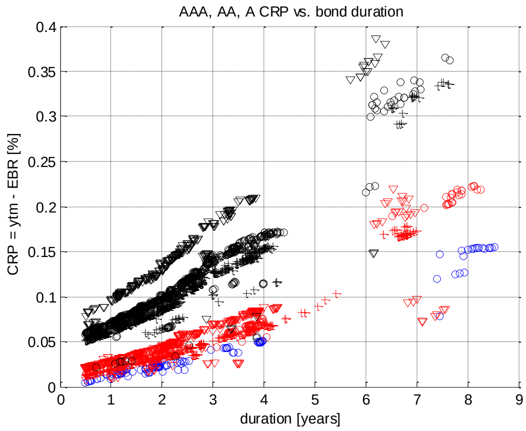


Figure 8b: "A" bond CRP versus duration

Comparing the term structure of bond spread to that of CRP, for the period September-December 2011, of straight corporate bonds S&P rated AAA (blue), AA (red), A (black). Where + denotes "+" rating, ○ denotes "pure" rating, and ∇ denotes "-" rating, including 164, 1,607, and 3,107 observations respectively. CRP calculations use S&P transition matrix (global data 1981-2010).

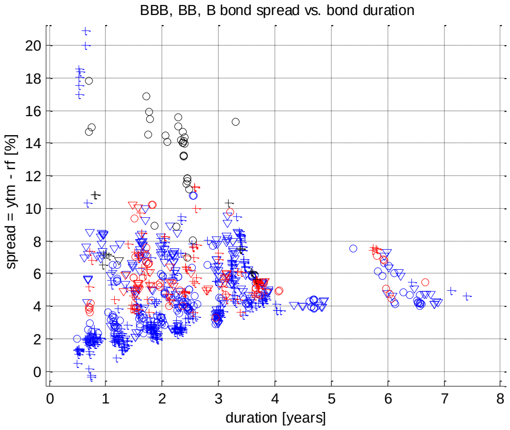


Figure 9a: “B” bond spread versus duration

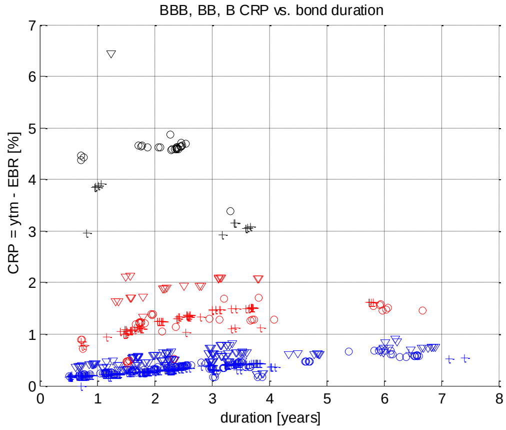
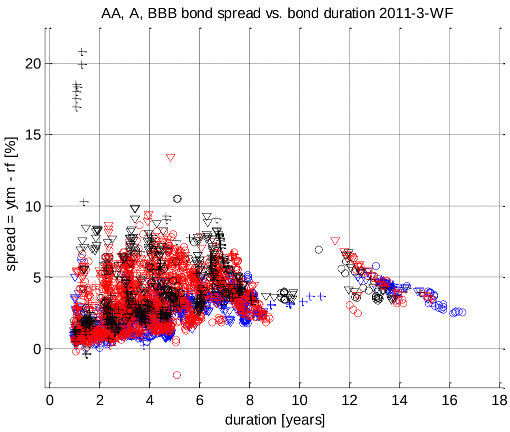


Figure 9b: “B” bond CRP versus duration

Comparing the term structure of bond spread to that of CRP, for the period September-December 2011, of straight corporate bonds S&P rated BBB (blue), BB (red), B (black). Where + denotes “+” rating, ○ denotes “pure” rating, and ∇ denotes “-” rating, including 901, 239, and 53 observations respectively. CRP calculations use S&P transition matrix (global data 1981-2010).



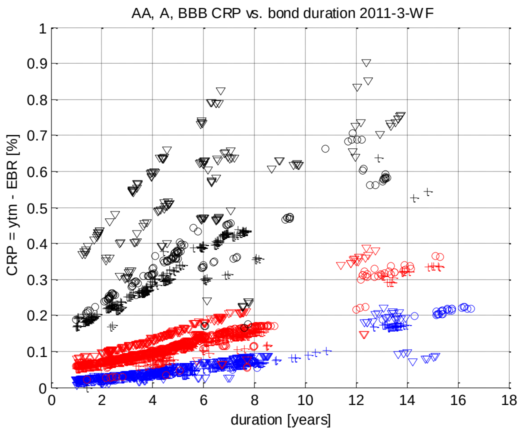
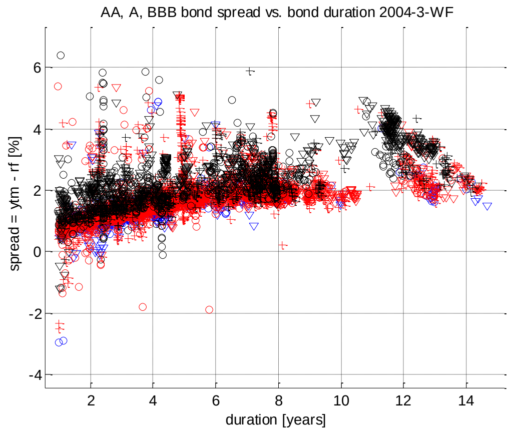


Figure 10a: "A,B" bond spread versus duration

Figure 10b: "A,B" bond CRP versus duration

Comparing the term structure of bond spread to that of CRP, for the period September-December 2011, of straight corporate bonds S&P rated AA (blue), A (red), BBB (black). Where + denotes "+" rating, ○ denotes "pure" rating, and ∇ denotes "-" rating, including 1,607, 3,107, and 901 observations respectively. CRP calculations use S&P transition matrix (global data 1981-2010). This figure is complementary to Figures 8 and 9, demonstrating the monotone increase of CRP between A rated and B rated bonds, delineating rating groups, whereas there bond spread are clearly mixed.



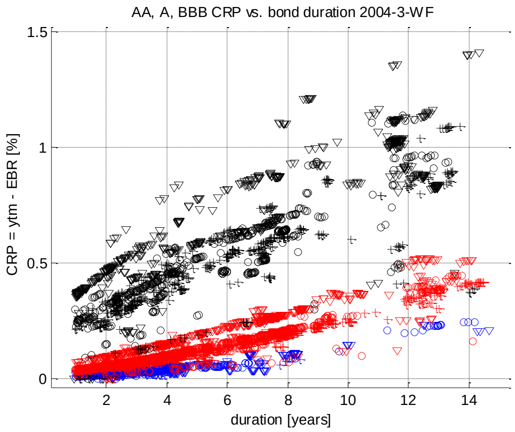


Figure 11a: "A,B" bond spread versus duration

Figure 11b: "A,B" bond CRP versus duration

Comparing the term structure of bond spread to that of CRP, for the period September-December 2004, of straight corporate bonds S&P rated AA (blue), A (red), BBB (black). Where + denotes "+" rating, ○ denotes "pure" rating, and ∇ denotes "-" rating, including 1,244, 6,025, and 2,769 observations respectively. CRP calculations use S&P transition matrix (global data 1981-2003). This figure demonstrates a similar behavior of observed bond spreads and CRP in 2004 compared to those of 2011 though both the spreads and CRP values are larger in 2011.

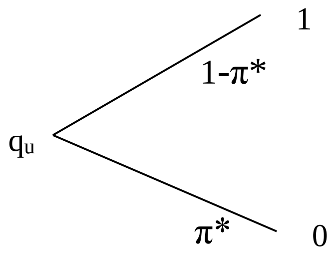


Figure 12a: security paying $1 at "up" state

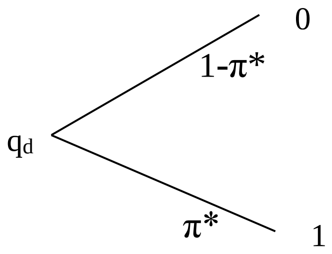


Figure 12b: security paying $1 at "down" state

q_u and q_d are the state prices of up (no default) and down (default) state respectively
π^* is the risk-neutral default probability

Appendix A: : List of Abbreviations

BS	bond spread
CDS	credit default swap
CEP	certainty equivalence premium
CRP	credit risk premium
CUSIP	Committee on Uniform Securities Identification Procedures
EBR	expected bond return
FISD	Fixed Income Securities Database
ISIN	International Securities Identification Number
JLT	Jarrow, Lando, and Turnbull
NASD	The National Association of Securities Dealers
NR	not rated
NS	Nelson-Siegel
RN	risk-neutral
S&P	Standard and Poor's
SIC	Standard Industrial Classification
TM	transition matrix
TRACE	Trade Reporting and Compliance Engine
TS	term structure
TSIR	term structure of interest rates
ytm	yield to maturity

Appendix B: Input Data Characteristics

Table B.1: sample size and 'SELL' type proportion by rating group

Sample size counts the number of observations in each rating group. Relative size is the proportion (in %) of observations in that group to the entire sample. Sell [%] is the proportion (in %) of sell observations in each rating group.

a. September-December 2004

Rating group	**Sample size**	**Relative size**	**Sell [%]**[(1)]
AAA	938	7.3	51.4
AA	1,244	9.6	47.8
A	6,025	46.6	48.0
BBB	2,769	21.4	52.4
BB	1,378	10.7	55.3
B	484	3.7	33.3
CCC/CC/C	96	0.74	54.2
D	2	0.02	100.0
All	12,936	100	49.4

b. September-December 2011

Rating group	**Sample size**	**Relative size**	**Sell [%]**[(1)]
AAA	164	2.7	50.6
AA	1,607	26.5	36.5
A	3,102	51.1	39.4
BBB	901	14.9	34.7
BB	239	3.9	26.8
B	53	2.7	56.6
All	6,066	100%	37.9

(1) Calculating sell percentage of transaction amount: $s/n = (1 - d/n)/2$ where s = sell amount, n = total amount (in the specific sample), and d is the difference of buy – sell transaction lines in the sample

Table b.2: Time to Maturity [years] Descriptive Statistics

a. September-December 2004

Rating group	**Observations**	**Mean**	**Std-dev**	**Median**	**Min**	**Max**
AAA	938	5.81	4.38	4.80	1.00	28.70
AA	1,244	4.86	4.14	4.12	1.02	29.12
A	6,025	6.69	4.90	5.45	1.00	29.99
BBB	2,769	8.32	7.66	5.23	1.01	30.00
BB	1,378	11.13	7.03	9.32	1.23	29.45
B	484	9.32	5.39	10.00	2.34	29.30
CCC/CC/C	96	6.79	6.98	4.73	1.02	25.26
D	2	13.26	15.88	13.26	2.04	24.49
All	12,936	7.37	6.03	5.52	1.00	30.00

a. September-December 2011

Rating group	**Observations**	**Mean**	**Std-dev**	**Median**	**Min**	**Max**
AAA	164	7.45	8.36	3.96	1.01	29.66
AA	1,607	6.44	5.95	4.83	1.00	29.99
A	3,102	5.58	3.93	4.77	1.01	29.82
BBB	901	6.51	4.97	5.01	1.01	29.52
BB	239	7.63	5.13	6.45	1.46	28.58
B	53	5.68	2.50	6.14	1.46	12.21
All	6,066	6.08	4.92	4.84	1.00	29.99

Table b.3: bond annual coupon rate [%] data summary by rating group

a. September-December 2004

Rating group	**Observations**	**Mean**	**Std-dev**	**Median**	**Min**	**Max**
AAA	938	4.76	1.17	4.95	2.00	8.31
AA	1,244	4.86	1.39	4.75	1.00	7.95
A	6,025	5.42	1.31	5.25	0.25	9.13
BBB	2,769	6.30	1.48	6.40	2.50	10.38
BB	1,378	8.59	1.56	8.25	4.75	14.50
B	484	7.17	2.11	6.88	5.00	13.63
CCC/CC/C	96	8.50	1.35	7.75	6.38	11.00
D	2	8.85	1.98	8.85	7.45	10.25
All	12,936	5.93	1.80	5.75	0.25	14.50

a. September-December 2011

Rating group	**Observations**	**Mean**	**Std-dev**	**Median**	**Min**	**Max**
AAA	164	3.07	1.32	2.95	0.50	6.00
AA	1,607	4.07	1.40	4.38	0.75	8.88
A	3,102	4.43	1.49	4.50	1.13	9.13
BBB	901	5.66	1.73	5.75	1.88	9.75
BB	239	6.84	1.47	6.38	3.88	12.00
B	53	6.94	1.21	6.90	5.40	10.75
All	6,066	4.59	1.68	4.75	0.50	12.00

Table b.4: Yield to Maturity [%] Descriptive Statistics

a. September-December 2004

Rating group	Observations	Mean	Std-dev	Median	Min	Max
AAA	938	3.96	0.81	3.97	2.24	7.85
AA	1,244	4.30	7.74	3.77	-15.87	NA*
A	6,025	4.54	4.51	4.29	-0.52	NA*
BBB	2,769	5.36	7.32	4.83	1.02	NA*
BB	1,378	7.67	1.80	7.48	3.04	23.77
B	484	7.65	2.59	6.89	4.12	17.31
CCC/CC/C	96	17.43	9.65	16.08	5.67	59.35
D	2	15.88	6.22	15.88	11.49	20.28
All	12,936	5.20	5.52	4.59	-15.87	NA*

a. September-December 2011

Rating group	Observations	Mean	Std-dev	Median	Min	Max
AAA	164	1.89	1.25	1.52	0.09	5.43
AA	1,607	2.97	1.45	2.80	-0.01	7.28
A	3,102	3.73	1.78	3.73	-1.45	13.78
BBB	901	4.87	4.71	4.31	-0.34	NA*
BB	239	6.05	1.53	5.70	3.49	11.56
B	53	11.87	3.47	12.01	5.92	17.82
All	6,066	3.81	2.63	3.60	-1.45	NA*

* erroneous data, excluded from the analysis (included here for completeness)

Appendix C: Discussion following Proposition 1

This appendix extends the discussion that follows Proposition 1.

Bond spread, risk-neutrality, and state prices

Fixed income literature and practitioners often use the term bond spread (BS) for the difference between a risky bond yield and the risk free rate: BS = $ytm - r$.[28] Since CEP and EBR are new terms, we discuss here their relation to the bond spread. For completeness we also develop the state prices of the default and non-default states.

Assume the prices of state contingent claims (Arrow-Debreu securities) are defined by Figures 12a and b:

[INSERT FIGURE 12a,b]

The law of one price requires:

(34) $$\frac{1}{R} = q_u + q_d \text{ and } p = q_u + \delta \cdot q_d .$$

From the prices of these two traded securities we can easily calculate the state prices:

(35) $$q_u = \frac{p - \delta / R}{1-\delta}, \qquad q_d = \frac{1}{R} - q_u = \frac{1/R - p}{1-\delta}.$$

The state prices are positive and well defined when $1/p > R > \delta/p$ and $\delta < 1$. The risk neutral probability of a "down" state is given by:

$$\pi^* = q_d \cdot R = \frac{1 - p \cdot R}{1-\delta} = \frac{1/p - R}{1/p - \delta/p} = \frac{u-R}{u-d}.$$

where: $u = 1/p$ and $d = \delta/p$ are the "up" and "down" gross returns of the risky bond.[29] Rearranging the above and using the definitions of *BS* and *yield reduction*:

(36) $$\pi^* = \frac{(1+ytm) - R}{\frac{1-\delta}{p}} = \frac{BS}{yield\ \ reduction}, \text{ hence } BS = E^*\left[yield\ reduction\right]$$

,

where E^* denotes expectations under the risk-neutral measure. Comparing *BS* with *CRP*, we see that both are expected values of the *yield reduction*, under the risk-neutral and physical probabilities respectively:

---

[28] This definition of BS requires a refinement when the term structure of the risk free rate is not flat, yet for our purpose in this section the simple definition suffices.

[29] These results and definitions are identical to those of binomial trees used in option pricing where $d < R < u$, otherwise admitting arbitrage opportunities. Thus π^* and 1- π^* are non-negative and smaller than 1, forming the RN probability set which is defined uniquely by the market price of the risky zero-coupon bond, its recovery rate and the market risk-free rate for the period.

(37) $$BS - CRP = E^{*}\left[yield\,reduction\right] - E\left[yield\,reduction\right] = \left(\pi^{*} - \pi\right)\left(1-\delta\right)/\,p$$

.

Rearranging (35) and using our definition of expected yield reduction:

$$CEP = \frac{1}{p} - R - \pi \cdot \frac{1-\delta}{p} = \frac{1}{p} - R - E[yield\;\;reduction],$$

$1/p - R$ is the bond spread (by definition) and equals the risk-neutral expected yield reduction by (36). Therefore:

(38) $$CEP = E^{*}[yield\;\;reduction] - E[yield\;\;reduction] = BS - CRP\,.$$

It is also easy to show that:

(39) $$CEP = \frac{1}{p} - \pi \cdot \frac{1-\delta}{p} - R = R\,\frac{(\pi^{*} - \pi)(1-\delta)}{1 - \pi^{*}(1-\delta)}\,.$$

This result shows clearly that CEP which embodies the risk aversion premium is monotonically increasing with the difference between the risk neutral and the physical probabilities of default. It is also directly (though not linearly) related to the relative loss given default (1-δ) on the bond.

CEP and systematic risk premium

The introduction mentions two approaches to expected return modeling and analysis, the risk approach and the cash-flow approach. The EBR in this paper can be classified to the latter. The CAPM is a cornerstone of the risk approach. We are not aware of an expected return risk model that includes promised payoffs and hence we believe there is no published analog to CRP in the risk approach literature. We focus our attention here on the CEP.

The time t value of an asset paying a cash-flow of CF(t+1) the next period is:

(40) $$BS - CRP = E^{*}\left[yield\,reduction\right] - E\left[yield\,reduction\right] = \left(\pi^{*} - \pi\right)\left(1-\delta\right)/\,p$$

,

where E_t is the expectation conditional on the information available at time t, r is the risk free rate, and γ is the risk premium demanded for bearing the risk of the cash-flow at time t+1. We further assume that CF(t+1) represents the entire value of the asset (including dividends, sales proceeds, etc.). All rates are on a per-period basis. Under the CAPM assumptions, the risk premium is:

(41) $$\gamma = \beta \cdot \left(E[r_m(t+1)] - r\right),$$

where we apply the usual notation of β and the return on the market portfolio m.

For a single period bond we use the following relations (by definition):

(42) $$p(t) = \frac{promised[CF(t+1)]}{1+y} = \frac{E_t[CF(t+1)]}{1+EBR}.$$

We explain above that in a frictionless world EBR differs from the risk-free rate by the certainty equivalence premium (CEP): EBR = r + CEP see equation (7).

CEP is required by investors for accepting the risky lottery whose uncertain payoff CF(t+1) are not guaranteed at the expected level of E[CF(t+1)], they might be higher or lower. In a risk-neutral world CEP = 0 whereas for risk averse investors CEP > 0.

Comparing the general case CAPM valuation of equations (40) and (41) with the specific definitions for a single period bond of equations (42) and (7) it is obvious that the risk premium required under CAPM is γ = CEP. Hence for a single period bond (zero coupon), under a frictionless market and the CAPM assumptions equation (43) holds:

(43) $$CEP = \beta \cdot \left(E[r_m] - r\right).$$

Outstanding CRP observations

In this paper we discuss the practical application of the model and present empirical results. Most of the CRP values follow a "well behaved" pattern and cluster along a term structure according to their credit ratings. Some observations however exhibit either a high or low CRP relative to their rating group. Here we look at potential explanations for such deviant observations.

This is not a regular sensitivity analysis. We assume here that there are two sets of similar, yet not identical data – one used by the modeler for the calculation, denoted by C ("calculation" values) and the other used collectively by the market, implicitly or explicitly, denoted by M. By construction, the price is identical in both sets, it is the market price. However, the values of recovery rates and of the default probabilities may defer.

When the two information sets are identical, except for the probability of default, following equation (9) the ΔCRP^{π}, difference between CRP(π_C) and CRP(π_M) (of the modeler and the market respectively) is expressed by equation (44) and the relative difference by equation (45).[30]

(44) $$\Delta CRP^{\pi} = CRP\left(\pi_C\right) - CRP\left(\pi_M\right) = \left(\pi_C - \pi_M\right)\left(\frac{1-\delta}{p}\right)$$

(45) $$\frac{\Delta CRP^{\pi}}{CRP\left(\pi_M\right)} = \frac{\pi_C}{\pi_M} - 1$$

Similarly, when the two information sets are identical, except for the recovery rate, the ΔCRP^{δ} difference is expressed by equation (46) and the relative difference by equation (47).

(46) $$\Delta CRP^{\delta} = CRP\left(\delta_C\right) - CRP\left(\delta_M\right) = -\frac{\pi}{p}\left(\delta_C - \delta_M\right)$$

[30] Although the price p depends on the risk neutral probability of default ($p = \left[1 - \pi_D^*\left(1-\delta\right)\right] / R$), and the risk-neutral the physical measures are dependent, the modeller uses the market price and thus equations (44) and (45) hold.

$$\frac{\Delta CRP^{\delta}}{CRP\left(\delta_M\right)} = \frac{1-\delta_C}{1-\delta_M} - 1 \qquad (47)$$

Hence, the modeler may calculate CRP values where these two effects are in the same direction, increasing or decreasing the calculated CRP, or where $\Delta \mathrm{CRP}^{\pi}$ and $\Delta \mathrm{CRP}^{\delta}$ have opposite signs.