HOMOGENEOUS OF DEGREE ONE GAMES ARE BALANCED WITH APPLICATIONS TO SERVICE SYSTEMS

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Homogeneous of degree one games are balanced with applications to service systems

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Abstract

A cooperative game with transferable utility is said to be *homogeneous of degree one* if for any integer m, the value of cloning m times all players at any given coalition, leads to m times the value of the original coalition. We show that this property coupled with sub-additivity, guarantee the non-emptyness of the core. A few examples for such games, which naturally emerge when servers in queueing systems cooperate, are presented.

1 Introduction

Service providers may benefit from cooperation among themselves, as, for example, in contact centers or airlines code sharing. The benefits are usually measured in the reduction of waiting time or in the reduction of the total service capacity needed in order to achieve the same performance measures. Towards the formation of this cooperation, service providers need to bargain, or even pay each other, in order to agree on how the reduced cost, or equivalently the gains due to cooperation, should be shared among themselves. Towards that end, the model of cooperative games with transferable utility is useful. In this model, each set of service providers is associated with a value, which is the cost induced when service providers of this set, and only this set, cooperate. Then, solution concepts, such as the Shapley value, may determine which share of the total cost resulting under full cooperation should be assigned to each service provider.

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Our main concern, in this paper, is whether the core of some games emerging from natural cooperation among service providers are non-empty. Note that a core allocation is a split of the total cost such that no subset of service providers can object on the ground that the total cost assigned to them is larger than the cost which they would have paid, if they broke away and cooperated only among themselves. Below we define a set of games, which are common when one looks into cooperation among service providers, and state what we prove to be a sufficient condition for the non-emptyness of the core. Thus, our paper deals first with games at large and only later exemplifies its usefulness when a couple of cooperation mechanisms among service providers are modeled.

A general cooperative game is defined by a set $N = \{1, 2, \ldots, n\}$ of n players. Any subset S of N, $\emptyset \subseteq S \subseteq N$, is called a *coalition.* For any coalition S a real value denoted by $V(S)$ is associated. This value represents the total cost inflicted on the members of coalition S when they cooperate. It is assumed that $V(\emptyset) = 0$. The function $V : \Re^{2^n-1} \rightarrow \Re$ is called a *characteristic function.* The pair of N and V is denoted by $G = (N, V)$ and it is called a *cooperative game with transferable utility.* The game is called *sub-additive* if for any two disjoint coalitions S and T, $V(S \cup T) \leq V(S) + V(T)$. It is clear that for such games, the socially best partition of the players of N to disjoint coalitions is when all players join the single large coalition, N itself, called the *grand coalition*, as the sum of the costs over the coalitions in any partition of N is minimized when the grand coalition is formed. In other words, sub-additive games call for the formation of the grand coalition, and therefore, the social cost inflicted is $V(N)$. Sub-additive games bear the concept of *economies of scope*: When each player, or set of players, contribute their own skills and resources, the total cost is less than the sum of the costs of the individual parts.

In order to guarantee the stability of the grand coalition, players need to agree on the split of the cost $V(N)$ among themselves. Here is where the details of the characteristic function play an important role. A cost allocation $(x_1, x_2, \ldots, x_n) \in \Re^n$ is called a *core allocation* if $\Sigma_{i=1}^{n} x_i = V(N)$ and if for any $\emptyset \subset S \subset N$, $\Sigma_{i \in S} x_i \leq V(S)$. Thus, the core is formulated as a linear programming formulation with n decision variables and $2^n - 1$ constraints. So searching the core by trial and error is practically almost impossible, except for specific problems having a special structure. This issue coupled with the possibility that the core is empty, makes the problem of finding a core allocation a real challenge as searching for a core cost allocation repeatedly by using a computerized procedure may be futilely. Thus, answering a-priori and affirmatively the question of the non-emptiness of the core, is a prominent first practical step in the process of the investigation of the core of a game.

In the sequel we focus on games $G = (N, V)$ whose characteristic function is sub-additive. There exist examples that show that sub-additivity by itself does not guarantee the non-emptiness of the core. In Section 2 we suggest what is, to the best of our knowledge, a novel condition on the characteristic function, which if met together with sub-additivity, guarantees the non-emptyness of the core. But before doing so, we review two known ways to establish the non-emptyness of the core. In Section 4 we present examples where these two ways are not helpful in determining whether the respective cores are non-empty, where the new proposed condition successfully unveils the vagueness upon this question.

- **Condition 1.** A game $G = (N, V)$ is said to be a *concave game* if its characteristic function is concave, meaning that for any two coalitions $S, T \subseteq N$, $V(S \cup T) + V(S \cap T) \leq V(S) + V(T)$. Clearly, concave games are sub-additive but not the other way around. It was shown in [15] that concave games have a non-empty core.

- **Condition 2.** The second way to establish the non-emptyness of the core is via showing that the game under consideration is a *market game*, see e.g., Chapter 13 in [13]. Market games refer to the special case where there are ℓ inputs and each of the n players possesses a commitment vector (endowments) $w_i \in \Re^\ell_+$, $1 \leq i \leq n$, which states a nonnegative value for each input. An *input vector* is a vector in $\Re^\ell_+$. Moreover, each player is associated with a continuous and convex cost function $f_i : \Re^\ell_+ \to \Re_+$, $1 \leq i \leq n$. A profile $(z_i)_{i\in N}$ of input vectors for which $\sum_{i\in N} z_i = \sum_{i\in N} w_i$ is an *allocation.* The game is such that a coalition S of players looks for an optimal way to redistribute its members' endowmets among its members in order to get a profile $(z_i)_{i\in S}$ of input vectors so as the sum of the costs across the members in S is minimized. Formally, for any $\emptyset \subseteq S \subseteq N$,

$$V(S) = \min\{\sum_{i\in S} f_i(z_i) : z_i \in \Re^\ell_+, i \in S \text{ and } \sum_{i\in S} z_i = \sum_{i\in S} w_i\} \tag{1}$$

  Market games are well-known to possess a non-empty core. Moreover, in [13], p. 267 a core allocation based on competitive equilibrium prices, is stated. Although highly related, our examples do not belong to this class of games.

Remark 1. In [13] it is assumed that the functions $f_i(z_i)$, $1 \le i \le n$, are non-increasing but as noted in [7] page 163 footnote 2, this in fact is not required.

In Section 2 we present a new property called *homogeneity of degree one*, and the key theorem of this paper, which states a sufficient condition for the non-emptiness of the core. In Section 3, the proof of the theorem is presented. We conclude with some examples in Section 4.

2 Homogeneity of degree one

Suppose we are given an infinite sequence of symmetric functions $V_0, V_1, V_2, \ldots$ where the input to the function V_m, $m \ge 1$, is m real vectors of size κ, for some given integer $\kappa \ge 1$, and the output is a real number. Let $V_0 \equiv 0$. Thus, for $m \ge 1$, $V_m : (\Re^\kappa)^m \to \Re$. Let $y^i \in \Re^\kappa$ for $i = 1, \ldots, m$ be an input to function V_m. Let $\phi(y^1, \ldots, y^m)$ be any permutation of $(y^1, \ldots, y^m)$. The symmetric property of the function V_m implies that $V_m(y^1, \ldots, y^m) = V_m(\phi(y^1, \ldots, y^m))$, meaning that the value returned by the function V_m is independent of the input order of the m vectors in $\Re^\kappa$ into V_m.

In the context of games, we assume that any potential player of the game is associated with a vector of properties in $\Re^\kappa$. In the market game presented at the end of Section 1, the vector of properties can be the commitment vector of a player. We assume that if m players form a coalition, then the cost of the coalition depends only on the size of the coalition, namely m, and the m vectors of properties associated with the members of the coalition. Note that this presentation refers to games where the set of potential players can be infinitely large (not necessarily countable). Using the conventional notation, for any given finite set of players $N = \{1, \ldots, n\}$, where each player $i \in N$ is associated with a vector of properties $y^i \in \Re^\kappa$, we associate the game $G = (N, V)$, which is defined by N and the characteristic function V, such that for any subset $\emptyset \subseteq S \subseteq N$, $V(S) = V_{|S|}((y^i)_{i \in S})$. Therefore, given the sequence of functions $\{V_m : m \ge 0\}$, one can define for any subset of players of N and their vectors of properties the associated game $G = (N, V)$. We call a game $G = (N, V)$ of the above described structure, a *regular game*.

Next we present a few examples of regular games, and one of a game which is not regular:

- **Example 1.** In the airport problem, N represents a set of airlines, and any coalition of airlines S, $S \subseteq N$, may share an airstrip. Each airline

$i \in N$ is associated with c_i, which is the cost of the airstrip that the airline needs. Serving airline i implies serving any airline with a cost parameter that is at most as large as c_i. Accordingly, a cooperative game (N, V) is defined such that for any $S \subseteq N$, $V(S) = \max_{i \in S} c_i$, see [5]. In this game, each airline is associated with a vector of properties of size 1, namely, $y^i = c_i$, and $V_m : \Re^m \to \Re$, where for a set S of m vectors of properties, $V_m((c_i)_{i \in S}) = \max_{i \in S} c_i$.

- **Example 2.** The second example is the cooperation in the queueing game considered in [3]. There, a set $N = \{1, \ldots, n\}$ of n $M/M/1$ queueing systems, cooperate in order to minimize the steady-state congestion in the combined system. Queueing system i is associated with its own exponential service rate μ_i and its own Poisson arrival rate of customers λ_i, $\lambda_i < \mu_i$, $i \in N$. Cooperation of a set $S \subseteq N$ in this model results in a single $M/M/1$ queue whose capacity is the sum of the capacities of the individual servers in S, and whose arrival rate is the sum of the individual arrival rates in S. For any coalition $S \subseteq N$ the congestion of S is thus given by

$$V(S) = \frac{\lambda(S)}{\mu(S) - \lambda(S)}, \tag{2}$$

where $\mu(S) = \sum_{i \in S} \mu_i$ and where $\lambda(S) = \sum_{i \in S} \lambda_i$. Therefore, in this case, each queueing system is associated with a vector of properties of size 2, namely $y^i = (\lambda_i, \mu_i)$ for $i \in N$, and $V_\ell : (\Re^2)^\ell \to \Re$, where for a set S of ℓ vectors of properties,

$$V_\ell((\lambda_i, \mu_i)_{i \in S}) = \frac{\sum_{i \in S} \lambda_i}{\sum_{i \in S}(\mu_i - \lambda_i)}. \tag{3}$$

This game was shown in [3] to have a non-empty core, although it is neither monotone nor concave.

- **Example 3.** The third example is the cost allocation problem for the first order interaction joint replenishment model, see [2]. In this model a set of retailers $N = \{1, \ldots, n\}$ orders stock from a single warehouse. Each retailer $i \in N$ faces a constant demand rate d_i, and it pays a linear inventory holding cost h_i per unit of stock per unit of time. The setup cost structure consists of minor setup costs, so that retailer $i \in N$ pays a minor setup cost K_i each time she places an order. In addition, there exists a major setup cost K_0, that is paid each time that at least one retailer places an order, so that if a group of retailers $S \subseteq N$ order simultaneously, the setup cost incurred at that time is $K_0 + \sum_{i\in S} K_i$. The optimal replenishment policy that minimizes the long-run average cost is unknown, yet [10] and [14] show that an optimal policy among the *power of two* policies comes within 2% of optimality. They also show how to calculate its corresponding cost. In this example, each retailer $i \in N$ is associated with a vector of properties that is a triplet $y^i = (K_i, h_i, d_i) \in \Re^3$. The function $V_\ell : (\Re^3)^\ell \to \Re$, is defined so that it returns the optimal cost of a power of two policy for any given ℓ vectors of properties for any $\ell \geq 1$. The characteristic function of the corresponding game (N, V) returns the optimal cost of a power of two policy for any subset of retailers $S \subseteq N$. This game was shown in [2] to have a non-empty core, in spite of the fact that it is not concave.

- **Example 4.** We conclude this list by an example, which is similar to the third one, where the joint setup cost structure for a group of retailers $S \subseteq N$ is given by a general submodular function $K(S)$, instead of the cost structure $K_0 + \sum_{i\in S} K_i$ considered in [2]. [16] considered the cost allocation problem of the joint replenishment model with submodular joint setup costs. In this model, each retailer is associated with a vector of properties of size 2, namely $y^i = (h_i, d_i)$, but the replenishment cost of a group $S \subseteq N$ of retailers does not necessarily have a closed-form expression as a function of $|S|$ and the $|S|$ vectors of properties of S, but it is some abstract function of S. The cost of a subset may depend, for example, also on the identity of its members. Therefore, this last game is not a regular game, as one cannot extend the definition of V beyond all subsets of N.

The above set of examples is by no means complete. Two more examples for regular games are stated in Section 4 below. See [11] for an additional example. Next, we broaden the definition of sub-additivity of a characteristic function in order that it will fit regular games $G = (N, V)$:

Definition 1 *The characteristic function of a regular game $G = (N, V)$, where each player $i \in N$ is associated with a vector of properties in $\Re^\kappa$, is said to be sub-additive if the corresponding sequence of symmetric functions $V_0, V_1, V_2, \ldots$, where $V_\ell : (\Re^\kappa)^\ell \to \Re$, $\ell \geq 1$, and $V_0 \equiv 0$, satisfies the following property: given two finite (not necessarily disjoint) sequences of vectors of properties in $\Re^\kappa$, $(y_A^i)|_{i\in A}$ and $(y_B^i)|_{i\in B}$, then $V_{(|A|+|B|)}((y_A^i)|_{i\in A}, (y_B^i)|_{i\in B}) \leq V_{|A|}((y_A^i)|_{i\in A}) + V_{|B|}((y_B^i)|_{i\in B})$.*

In order to present the sufficient condition for the non-emptiness of the core of a regular game $G = (N, V)$, we need to apply functions $V_{\ell n}$ on ℓ

copies (duplicates) of each vector of properties y^i, $i \in N$, where $\ell = 1, 2, \ldots$. For this sake, for any integer $j \geq 1$, let the set $N^j = \{(1, j), (2, j), \ldots, (n, j)\}$ consists of the $j - th$ copy of the set of players N, where player $(i, j) \in N^j$ is associated with the vector of properties $y^i_j = y^i \in \Re^\kappa$, $i \in N$. Let $N^{(m)} = \cup_{j=1}^m N^j$. Thus, $N^{(m)}$ contains nm elements, and the infinite sequence of sets $N^{(1)}, N^{(2)}, \ldots$, is nested. We define, similarly, S^m and $S^{(m)}$ for any coalition $S \subset N$. We also let $G^m = (N^{(m)}, V)$, where $G^1 = G$, be a sequence of games, so that the definition of V in game G^m is the extension of the definition of V for game G^1 to any coalition T, $T \subseteq N^{(m)}$. This extension is straightforward as $G = (N, V)$ is a regular game.

We next define the new proposed property of regular games, i.e., homogeneity of degree one, and then we state the main theorem of this paper.

Definition 2 *The characteristic function V of a regular game $G = (N, V)$, is homogeneous of degree one if the corresponding sequence of symmetric functions $V_1, V_2, \ldots$ satisfies that for any integer $m \geq 1$ and any subset $S \subseteq N$, $V_{|S|m}((y^i_j)_{(i,j)\in S^{(m)}}) = mV(S)$, or equivalently, game $G^m = (N^{(m)}, V)$, satisfies $V(S^{(m)}) = mV(S^{(1)})$.*

Remark 2. Homogeneity of degree one means that when two (or more) identical sets of players cooperate, they cannot do better than they did when acting individually. At the same time, they do not interfere each other. What they produce is just the total of what they would have done separately. This in fact means lack of economies of scale. Note that sub-additivity means that gains due to cooperation are possible. This coupled with homogeneity of degree one mean that in order to get strict improvement due to cooperation, the cooperating sets should be different, i.e., at least one of the cooperating subsets should contain types of players that do not appear in the other set.

Next we state the main result of this paper.

Theorem 1 *If the characteristic function V of a regular game $G = (N, V)$ is sub-additive and homogeneous of degree one, then the core of the game is non-empty.*

Example 5. Suppose each player $i \in N$ is associated with a positive number a_i. Define the game $G = (N, V)$ by $V(S) = |S| \min_{i\in S} a_i$. It is easy to see that this game is regular, sub-additive and homogeneous of degree one (but not concave). Hence, by Theorem 1, it possesses a non-empty core. A core allocation is $x_i = \min_{j\in N} a_j$ for all $i \in N$. The core contains infinitely many allocations, unless $a_i = a$, $i \in N$, for some constant a. In the latter case the core is a singleton: $x_i = a$, $1 \leq i \leq n$.

We like to note that a related result for non-atomic games, appears in [4], p. 167.[1] There it says that for a sub-class of these games (denoted there by pNA), sub-additivity and homogeneity of degree one (defined in a way suitable for non-atomic games) guarantee the non-emptiness of the core. Indeed, they proved that the core is a singleton that coincides with the Aumann-Shapley prices.

In Section 3 we prove Theorem 1. To indicate the usefulness of the theorem, we present in Section 4 two examples of regular games whose characteristic function is sub-additive and homogeneous of degree one. These two examples do not belong to any of the two abovementioned types of games (concave games or market games) that guarantee the non-emptiness of the core. The two examples are based on queueing models in which the cost of a coalition is a function of the steady-state congestion, and cooperation among servers lead to reduction in this cost. In particular, there is a need to share the gains due to cooperation between the contributing servers.

3 Proof of Theorem 1

We start by reviewing a well known necessary and sufficient condition for the non-emptiness of the core of a cooperative game, see, e.g., [13], Chapter 13. This condition is equivalent to the duality condition of a feasible linear programming formulation. Specifically, let $\mathcal{C}$ be the set of all 2^n coalitions of N. For any coalition S denote by $\Re^S$, the $|S|$-dimensional Euclidean space in which the dimensions are indexed by the members of S, and denote by $1_S \in R^n$ the characteristic vector of S given by

$$(1_S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

Definition 3 *A collection $(\alpha_S)_{S \in \mathcal{C}}$ of numbers in $[0, 1]$ is said to be a balanced collection of weights if for every player $i \in S$ the sum of α_S over all coalitions that contain i equals 1, namely $\Sigma_{S \ni i} \alpha_S = 1$ for all $i \in N$. A coalitional game $G = (N, V)$ is said to be balanced if $\sum_{S \in \mathcal{C}} \alpha_S V(S) \geq V(N)$, for every balanced collection of weights $(\alpha_S)_{S \in \mathcal{C}}$.*

The following proposition is referred to as the Bondareva-Shapley Theorem, see, e.g., Proposition 262.1 in [13]:

[1] Non-atomic games are games in which each individual player contributes infinitesimally to the value of a coalition he joins.

Proposition 1 *A coalitional game with transferrable utility has a nonempty core if and only if it is balanced.*

We are now ready to prove Theorem 1.

Proof: We prove the theorem by using Proposition 1 in two steps. We first prove that for any vector of balanced rational weights $(\alpha_S)_{S\in\mathcal{C}}$, the inequality $\sum_{S\in\mathcal{C}}\alpha_S V(S)\geq V(N)$, holds. Then we prove that the same is the case for any balanced collection of real weights.

Consider any balanced collection of rational weights $(\alpha_S)_{S\in\mathcal{C}}$. Let $M(\alpha)$ be a positive integer such that $\tau_S(\alpha)=M(\alpha)\alpha_S$ is an integer for all coalitions $S\in\mathcal{C}$. As the game $G=(N,V)$ is regular, there exists an integer $\kappa\geq 0$, such that each member $i\in N$ is associated with a vector of properties $y^i\in\Re^\kappa$. Let $y^i_j=y^i$ for any integer $j\geq 1$. Regularity of the game implies that $V(S)=V_{|S|}((y^i)|_{i\in S})$. As V is homogenous of degree one, $V_{\tau_S(\alpha)|S|}((y^i_j)_{(i,j)\in S^{(\tau_S(\alpha))}})=\tau_S(\alpha)V(S)$. Note that

$$\sum_{S\in\mathcal{C}}\tau_S(\alpha)V(S)=\sum_{S\in\mathcal{C}}V_{\tau_S(\alpha)|S|}((y^i_j)_{(i,j)\in S^{(\tau_S(\alpha))}})\geq V_{M(\alpha)n}((y^i_j)_{(i,j)\in N^{(M(\alpha))}})=M(\alpha)V(N),$$

where the above inequality follows by the sub-additivity of V in the regular game $G=(N,V)$, and specifically, sub-additivity of V over $N^{(M(\alpha))}$ that contains $M(\alpha)$ repetitions of each player of N. Consider now the l.h.s. of the inequality, i.e., $\sum_{S\in\mathcal{C}}V_{\tau_S(\alpha)|S|}((y^i_j)_{(i,j)\in S^{(\tau_S(\alpha))}})$: for any $i\in N$, we have also here $\sum_{S\in\mathcal{C}:\ i\in S}\tau_S(\alpha)=M(\alpha)\sum_{S\in\mathcal{C}:\ i\in S}\alpha_S=M(\alpha)$ copies of each vector of properties y^i, as $(\alpha_S)_{S\in\mathcal{C}}$ is a balanced collection of weights. The last equation follows from the fact that in the regular game $G=(N,V)$, the characteristic function V is homogenous of degree one. To conclude, $\sum_{S\in\mathcal{C}}\tau_S(\alpha)V(S)\geq M(\alpha)V(N)$. Recall that $\tau_S(\alpha)=M(\alpha)\alpha_S$, thus dividing the last inequality by $M(\alpha)$ gives the desired result for any rational balanced collection of weights $(\alpha_S)_{S\in\mathcal{C}}$.

In order to complete the proof, we need to show that the above property holds also for any vector of balanced real weights. Let $\widetilde{(\alpha_S)}_{S\in\mathcal{C}}$, be a balanced collection of real weights. Consider the simplex induced by the constraints that define the set of balanced weights, i.e., $(\alpha_S)_{S\in\mathcal{C}}\geq 0$, and $\sum_{S\in\mathcal{C},\ i\in S}\alpha_S=1$ for all $i\in N$. The extreme points of this simplex are rational, as the righthand side of the constraints as well as the coefficients of the variables are 0 or 1. Let K be the number of extreme points of this simplex, and let α^j for $j=1,\ldots,K$, be the respective extreme points, where each α^j is a vector of size $|\mathcal{C}|$. Thus, $\widetilde{(\alpha_S)}_{S\in\mathcal{C}}$, can be represented as a convex combination of the extreme points: Let $(\gamma_1,\ldots,\gamma_K)$ be the respective weights, so that $0\leq\gamma_i\leq 1$ for $i=1,\ldots,K$, $\sum_{j=1}^K\gamma_j=1$, and

$\widetilde{(\alpha_S)}_{S\in\mathcal{C}} = \sum_{j=1}^{K}\gamma_j(\alpha_S^j)_{S\in\mathcal{C}}$. As each of the extreme points of the simplex is rational and is a vector of balanced weights, we have $\sum_{S\in\mathcal{C}}\alpha_S^jV(S) \geq V(N)$ for all $1 \leq j \leq K$. Therefore, $\sum_{S\in\mathcal{C}}\widetilde{(\alpha_S)}V(S) = \sum_{S\in\mathcal{C}}\sum_{j=1}^{K}\gamma_j\alpha_S^jV(S) = \sum_{j=1}^{K}\gamma_j\sum_{S\in\mathcal{C}}\alpha_S^jV(S) \geq \sum_{j=1}^{K}\gamma_jV(N) = V(N)$. ■

4 Examples - Cooperation in Queueing Systems

In [3] we dealt with what seems to be the simplest (but most revealing) possible model of cooperation in service systems. This model was described as Example 2 in Section 2 and is based on the assumption that when a set of servers cooperate, they work as a single server whose service rate is the sum of the individual service rates. Moreover, this combined server serves their joint stream of arrivals. In [3] it was shown that this game, in spite of not being concave, has a non-empty core. As mentioned in Section 2, this game is regular. However, it is easy to check that its characteristic function (3) is not homogenous of degree one. Indeed, under this cooperation there is both economies of scope as well as economies of scale.[2] See [17] for a related result. Note that there is also a vast literature on competition among servers, leading to non-cooperative game modeling. See for example [8] or [1].

In the following two subsections we present two examples for different (and more involved) cooperation among servers, for which Theorem 1 is useful in showing that the cores of the corresponding games are not empty.

4.1 Optimal Unobservable Routing for a Given Set of Servers

In this section we consider a cooperative queueing game defined by a set $N = \{1, 2, \ldots, n\}$ of servers, where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, are the given exponential capacities of the servers, and the servers who form a coalition have yet to work individually. Each of the servers in N is also associated with a Poisson arrival stream of customers, so that server $i \in N$ is associated with an arrival rate of λ_i, $\lambda_i < \mu_i$. We assume that all servers provide the same type of service, and that all customers require this type of service. When forming a coalition $S \subseteq N$ of servers, the central planner of the coalition can decide how to re-route the customers, namely which portion of them to assign to each of the servers. We also assume that the central planner has the option to outsource some (or all) of the customers to an external service provider at a fix cost per unit rate outsourced. The objective of the coalition's central planner is to find the optimal routing of the Poisson arrival rate $\lambda(S) =$

[2] Here when two identical coalitions cooperate the total costs is reduced by half.

$\sum_{i \in S} \lambda_i$ among the servers of S and the external service provider, so as that the total cost of the steady-state congestion at the servers of S plus the outsourcing cost, is minimized. The congestion is measured by the total of the mean steady-state number of customers in the system under the optimal split of the stream of arrivals to the servers. For simplicity, we assume that the cost per unit of congestion is one, and accordingly we normalize the fix cost per unit rate outsourced to b. For simplicity, we consider below the grand coalition, but the same considerations apply verbatim to any coalition $S \subset N$.

Given the grand coalition N, let Λ, $0 \leq \Lambda \leq \lambda(N)$, be the arrival rate of customers that is served internally, i.e., by the servers of N. For any given $\Lambda \leq \lambda(N)$, let the function $C(N, \Lambda)$ be the optimal steady-state congestion cost associated with N, given a Poisson arrival rate of Λ that is served by the servers of N under the optimal routing. For an insourced arrival rate of $\Lambda \leq \lambda(N)$, let $\lambda^I_{N,i}(\Lambda)$ denote the optimal insourced arrival rate to server $i \in N$ according to the cost function $C(N, \Lambda)$. Thus, for any $i \in N$, $\lambda^I_{N,i}(\Lambda) < \mu_i$ and $\sum_{i \in N} \lambda^I_{N,i}(\Lambda) = \Lambda$. We consider the cooperative game $G = (N, V^b)$ where the characteristic function $V^b(N)$ (and similarly $V^b(S)$ for any $\emptyset \subseteq S \subseteq N$) is defined as

$$V^b(N) = \min\{C(N, \Lambda) + b(\lambda(N) - \Lambda) : \ 0 \leq \Lambda \leq \lambda(N)\} \tag{4}$$

The characteristic function V^b clearly satisfies $V^b(\emptyset) = 0$ and it is sub-additive, as combining two disjoint sub-coalitions of N into one may only reduce the total cost of the steady-state congestion and outsourcing as more options for the optimal cooperation exist.

We start by analyzing $C(S, \Lambda)$. The optimal routing of the arrival rate Λ that achieves the minimum steady-state congestion may result in shutting off some of the slow servers. According to [6], $C(S, \Lambda)$ can be formulated as follows:

$$C(S, \Lambda) = \min\{\sum_{j \in S} \frac{\lambda^I_{S,j}(\Lambda)}{\mu_j - \lambda^I_{S,j}(\Lambda)} : \sum_{j \in S} \lambda^I_{S,j}(\Lambda) = \Lambda \text{ and } 0 \leq \lambda^I_{S,j}(\Lambda) < \mu_j \ for\ j \in S\} \tag{5}$$

In [6] it is shown that in an optimal solution to (5), only servers with a sufficiently high capacity are utilized. More precisely, for $\Lambda < \mu(S)$, we let $WRK(S, \Lambda) = \{j \in S : \ j \leq i^*(S, \Lambda)\}$ be the set of open servers in S that serve an arrival rate of Λ. The other servers, i.e., the servers in $\{j \in S : \ j > i^*(S, \Lambda)\}$ are closed. Note that $WRK(S, \Lambda)$ is never empty, where its complement set in S, may be empty if all servers are open. We define for convenience server $n+1$, with $\lambda_{n+1} = \mu_{n+1} = 0$. Let $i^+(S,\Lambda)$ be the server following $i^*(S,\Lambda)$ in S. If $i^*(S,\Lambda)$ is the last server in S, then $i^+(S,\Lambda)$ is defined as $n+1$. In general we use i^+ to be the server following i in the set under consideration. As was proved in [6]

$$i^*(S,\Lambda) = \min\left\{ i \in S : \quad \mu_{i^+} \leq \frac{(\sum_{j\in S, j\leq i} \mu_j - \Lambda)^2}{(\sum_{j\in S, j\leq i} \sqrt{\mu_j})^2} \right\} . \tag{6}$$

From [6], we can deduce that

$$C(S,\Lambda) = \frac{(\sum_{i\in S,\ i\leq i^*(S,\Lambda)} \sqrt{\mu_i})^2}{\sum_{i\in S,\ i\leq i^*(S,\Lambda)} \mu_i - \Lambda} - |WRK(S,\Lambda)| . \tag{7}$$

According to [6] the optimal rate of arrival to open server $k \in S$, $k \leq i^*(S,\Lambda)$, given a total insourced rate Λ, is

$$\lambda^I_{S,k}(\Lambda) = \mu_k - (\sum_{i\in S;\ i\leq i^*(S,\Lambda)} \mu_i - \Lambda) \frac{\sqrt{\mu_k}}{\sum_{i\in S\ i\leq i^*(S,\Lambda)} \sqrt{\mu_i}}. \tag{8}$$

Further details on this model can be found in [6], [8], p.64 or [9].

Back to the game $G = (N, V^b)$, where outsourcing at a cost b per unit rate outsourced is possible. If b is large enough then no service is outsourced, and the game $G = (N, V^b)$ coincides with the game (N, C), where for any coalition $S \subseteq N$, $V^b(S) = C(S, \lambda(S))$. At the other extreme, if the outsourcing cost is sufficiently small, all service is outsourced. More precisely, if $b \leq \mu_1^{-1}$, then $V^b(S) = b\lambda(S)$ for any $S \subseteq N$, and the game boils down to the trivial cooperative game where the characteristic function V^b is linear in the total stream of arrivals, namely $V^b(S) = b\lambda(S)$, resulting in a single core cost allocation $x_i = b\lambda_i$ for $i = 1, \ldots, n$.

For $i \in N$, let $f_i(z_i)$ be the congestion cost at server i due to an arrival stream whose rate is z_i and let $f_0(z_0)$ be the outsourcing cost for a rate z_0 of customers outsourced. I.e.,

$$f_i(z_i) = \frac{z_i}{\mu_i - z_i}, \quad 1 \leq i \leq n \tag{9}$$

and

$$f_0(z_0) = bz_0. \tag{10}$$

Thus,

$$V^b(N) = \min\{\sum_{i=0}^{n} f_i(z_i) : \ s.t. \ \sum_{i=0}^{n} z_i = \lambda(N)\}. \tag{11}$$

Let $\lambda_{N,i}$ denote the optimal insourced arrival rate allocated to server i, for $i \in N$. Also, let $\lambda_{N,0}$ be the optimal arrival rate outsourced. Clearly, $\sum_{i=0}^{n} \lambda_{N,i} = \lambda(N)$. Thus, $V^b(N) = C(N, \sum_{i=1}^{n} \lambda_{N,i}) + b\lambda_{N,0} = C(N, \lambda(N) - \lambda_{N,0}) + b\lambda_{N,0}$. Also, for $i \in N$, $\lambda_{N,i} = \lambda^I_{N,i}(\lambda(N) - \lambda_{N,0})$.

Next we present a general optimization result that will be useful in solving $V^b(N)$ and giving it an explicit presentation. The first item of the next lemma was proved in [18]. The second item is a slight generalization of the first, which fits the structure of problem (11).

Lemma 1 *Consider problem (P):*

$$\min\{\sum_{k=1}^{m} g_k(y_k) : \sum_{k=1}^{m} y_k = \Theta, \quad y_k \geq 0, \quad 1 \leq k \leq m\} \tag{12}$$

where $\Theta > 0$ *and* $g_k : \Re \rightarrow \Re$, *for* $1 \leq k \leq m$, *are strictly convex and continuously differentiable. Suppose also that the functions are ranked in a non-decreasing order of their partial derivatives at 0, i.e., the sequence* $\{\frac{dg_k(y)}{dy}|_{y=0^+}\}_{k=1}^{m}$ *is non-decreasing in* k. *Let* $q(\Theta)$ *be the optimal value of (P) as a function of* Θ.

- *Then, there exists a unique optimal solution* $y^* \in \Re^m$ *to (P), an integer* $K \leq m$, *and a unique real number* ν, *such that for* $k \leq K$, $y_k^* > 0$ *and* $\{\frac{dg_k(y)}{dy}|_{y=y_k^*}\} = \nu$, *and for* $K < k \leq m$, $y_k^* = 0$ *and* $\{\frac{dg_k(y)}{dy}|_{y=0^+}\} \geq \nu$. *Moreover,* $q(\Theta)$ *is strictly covex and continuously differentiable with* $\frac{dq}{d\theta}|_{\theta=\Theta} = \nu$.

- *Consider now a problem that is identical to problem (P) except for the fact that one of the* $g_k(\cdot)$ *functions is linear in* y_k *as opposed to being strictly convex. Then, the structure of the optimal solution described in the first item still holds, except that* $q(\Theta)$ *is now convex (as opposed to being strictly convex) and continuously differentiable. More specifically, there exists a constant* $\bar{\theta} \geq 0$ *such that* $q(\Theta)$ *is strictly convex on* $(0, \bar{\theta})$ *and linear for* $\Theta \geq \bar{\theta}$. *In the later range, the slope coincides with that of the* g_k *which is linear.*

Proof:

See [18] for the proof of the first item. It is straightforward to generalize the proof to the case that one of the m functions is linear, where the others are strictly convex. The uniqueness of the optimal vector y^* follows from the strict convexity of the $m-1$ functions. If the linear function is g_1 then

$q(\Theta) = g_1(\Theta)$. Otherwise, there exists $\bar{\theta} > 0$, so that $q(\Theta)$ is strictly convex on $(0, \bar{\theta})$, and thereafter $q(\Theta)$ is linear in Θ. $\blacksquare$

We are now ready to investigate the structure of the solution of $V^b(N)$. In order to show the dependence of $V^b(N)$ on the total arrival rate, let $V^b(N, \Upsilon)$ denote the optimal congestion plus outsourcing cost of a Poisson arrival stream rate of Υ. Thus, $V^b(N, \lambda(N)) = V^b(N)$. For simplicity, let $i^*(N) = i^*(N, \lambda(N))$. Let also

$$p(b) = \max\{k : k \geq 0,\ \mu_k^{-1} < b\} \tag{13}$$

$$\alpha(N) = \frac{(\sum_{k=1}^{i^*(N)} \sqrt{\mu_k})^2}{(\sum_{k=1}^{i^*(N)} \mu_k - \lambda(N))^2} \tag{14}$$

Lemma 2
- *Let $q(\Upsilon) = V^b(N, \Upsilon)$. The function $q(\Upsilon)$ is convex and continuously differentiable in $\Upsilon > 0$. Moreover, for any given value of Υ, the solution to the optimization function $V^b(N, \Upsilon)$ is unique.*

- *If $p(b) = 0$, (or equivalently, $b \leq \mu_1^{-1}$), then $V^b(N) = b\lambda(N)$, and the core of the cooperative game (N, V^b) is a singleton of the form $x_i = b\lambda_i$ for $i = 1, \ldots, n$.*

- *If $b \geq \alpha(N)$, then $\lambda_{N,0} = 0$, and $V^b(N) = C(N, \lambda(N))$, implying that $\lambda_{N,i} = \lambda^I_{N,i}(\lambda(N))$, (see (8)), for $i \in N$. Moreover, for $1 \leq i \leq i^*(N)$, $\frac{df_i}{d\lambda}|_{\lambda=\lambda_{N,i}} = \frac{\mu_i}{(\mu_i - \lambda_{N,i})^2} = \alpha(N)$, and for $i^*(N) < i \leq n$, $\frac{df_i}{d\lambda}|_{\lambda=0^+} = \frac{1}{\mu_i} \geq \alpha(N)$.*

- *If $\mu_1^{-1} < b < \alpha(N)$, then the set of open servers is $\{1, \ldots, p(b)\}$ (see (13)). Moreover, for $1 \leq i \leq p(b)$*

$$\lambda_{N,i} = \mu_i - \frac{\sqrt{\mu_i}}{\sqrt{b}}, \tag{15}$$

and $\frac{df_i}{d\lambda}|_{\lambda=\lambda_{N,i}} = b$. For $p(b) < i \leq n$, $\lambda_{N,i} = 0$, and $\frac{df_i}{d\lambda}|_{\lambda=0^+} = \frac{1}{\mu_i} \geq b$. Also, $\lambda_{N,0} = \lambda(N) - \sum_{i=1}^{p(b)} \lambda_{N,i}$. Finally,

$$V^b(N) = 2\sqrt{b}\sum_{i=1}^{p(b)} \sqrt{\mu_i} + b(\lambda(N) - \sum_{i=1}^{p(b)} \mu_i) - p(b). \tag{16}$$

Proof:

- The proof follows from the second item of Lemma 1.

- Note that $\frac{df_i(\lambda)}{d\lambda}|_{\lambda=0^+} = \mu_i^{-1}$ for $i \in N$. The sequence $\{\mu_i^{-1}\}_{i\in N}$ is non-decreasing in i, and under the assumption in this item, it is bounded from below by b. The only possible structure of a feasible solution that satisfies the structure of the optimal solution, as detailed in Lemma 1 item 2, is $\lambda_{N,i} = 0$ for $i \in N$, and $\lambda_{N,0} = \lambda(N)$. That means, that in this case, all customers are outsourced and $V^b(N) = b\lambda(N)$. Moreover, $V^b(S) = b\lambda(S)$. The rest of the claim follows immediately.

- The proof of this item follows from [6] and also from the proof of the second item in Lemma 1, as the proposed solution is the only solution that satisfies the properties of the optimal solution.

- Follows from the second item of Lemma 1.

■

Lemma 2 implies that for a total arrival rate of $\lambda(N)$, there exist two positive constants $b_1 < b_2$, ($b_1 = \mu_1^{-1}$, and $b_2 = \alpha(N)$), such that, if the outsourcing cost $b \in [0, b_1]$ then all service is outsourced, if $b \geq b_2$ then all service is insourced and provided by servers $\{1, \ldots, i^*(N)\}$, and otherwise some customers are insourced and served by servers $\{1, \ldots, p(b)\}$, $p(b) \leq i^*(N)$, and the rest is outsourced.

Having described the solution to $V^b(N)$, defined via (4), we pose the question of how to allocate the cost $V^b(N)$ among the servers of N in the cooperative game $G = (N, V^b)$. As stated in Section 1, we focus here on core allocations. We will show first that the two existing criteria described in the introduction (that may help in determining whether the core is non-empty) are not helpful when considering this game. The next example shows that the characteristic function $V^b(N)$ is not concave.

Example 6. Consider $N = \{1, 2, 3\}$, with $\mu_1 = \mu_2 = 100$, $\lambda_1 = \lambda_2 = 1$, $\mu_3 = 1$ and $\lambda_3 = 0.99$. Assume that b is very large, i.e. $b \geq \alpha(N)$, so outsourcing is not profitable. Let $S = \{1, 3\}$ and $T = \{2, 3\}$. We have here $V^b(\{1\}) = V^b(\{2\}) = 1/(100-1) = 0.01$, $V^b(\{3\}) = 0.99/(1-0.99) = 99$. In coalition S only server 1 is open and likewise server 2 in coalition T. Thus, $V^b(S) = V^b(T) = (1 + 0.99)/(100 - 1 - 0.99) = 0.02$. In coalition $S \cup T$ only servers 1 and 2 are open, each getting half of the total traffic. Hence, $V^b(S \cup T) = 2(\frac{1+0.495}{100-1-0.495})$. It is easy to see that the cost of the intersection $V^b(S \cap T) = V^b(\{3\}) = \frac{0.99}{1-0.99} = 99$. Hence, $V^b(S \cup T) + V^b(S \cap T) > V^b(S) + V^b(T)$, proving that the set function $V^b(\cdot)$ is not concave.

Remark 3. Note that similarly to the characteristic function dealt with in [3], $V^b(\cdot)$ is not monotone: for $S \subset T$, $V^b(S)$ can be equal, strictly smaller or strictly larger than $V^b(T)$.

Next we verify whether the game $G = (N, V^b(N))$ satisfies the second criteria, i.e., whether it is a market game. Indeed, the structure of the game resembles a market game, but as we are going to see, it is a market game only in the case where $b \geq \alpha(N)$, namely when the outsourcing option is not exercised. In such a case the input of each server $i \in N$ is λ_i and $f_i(z_i) = \frac{z_i}{\mu_i - z_i}$ which is continuous in $z_i < \mu_i$, non-decreasing and convex. Therefore, in the case of a game with no outsourcing, it is a market game and its core is non-empty. Otherwise, if $b < \alpha(N)$, then not all the total input of $\lambda(N)$ is distributed among the servers in N, and therefore it is not a market game. Note that we cannot consider the outsourcing option as an additional player of the game. Doing so would have made it a market game with $n+1$ players, but in this problem the total cost should be allocated only among the n servers of N.

We next resort to the new proposed criterion presented in Section 2. First, we note that the game $G = (N, V^b)$ is regular as each server $i \in N$ is associated with a vector of input properties of size 2, namely (λ_i, μ_i), where $\lambda_i < \mu_i$, and the cost $V^b(N)$ (and similarly, $V^b(S)$ for any $S \subseteq N$) defined through (9)-(11), depends only on these input vectors. Thus, we can define an infinite sequence of symmetric functions $V_0, V_1, V_2 \ldots$, where $V_0 \equiv 0$, so that the input to V_m is m non-negative vectors of size 2 whose first component is smaller than the second one, and where V_m maps $(\Re^2_+)^m \to \Re$. The sub-additivity of the characteristic function $V^b(S)$, $\emptyset \subseteq S \subseteq N$ follows directly as the optimal cost of the union of two disjoint sets of players can not increase above the sum of the original costs. In order to conclude the proof of the non-emptiness of the core, it only remains to show that the characteristic function V^b is homogenuous of degree 1.

Lemma 3 *The characteristic functions $V^b(\cdot)$ for any $b \geq 0$ defined in (11) is homogenous of degree 1.*

Proof: In $N^{(m)}$ each of the m copies of server $i \in N$, namely servers $(i,1), \ldots, (i,m)$, is associated with capacity μ_i, rate of arrival λ_i, and the function f_i given in (9), which represents the congestion cost at server (i,j), $1 \leq j \leq m$. Index the nm servers in $N^{(m)}$ is a non-increasing order of their capacities. Finally, let f_0 be defined as in (10). We can write $V^b(N^{(m)})$ as follows:

$$V^b(N^{(m)}) = \min\{ \sum_{k \in \{0\} \cup N^{(m)}} f_k(\lambda_k) : \quad \text{s.t.} \sum_{k \in \{0\} \cup N^{(m)}} \lambda_k = m\lambda(N)\}.$$

As the functions $f_k(\cdot)$ for $k = 1, \ldots, nm$, are strictly convex, and $f_0(\cdot)$ is linear, the structure of the solution for $V^b(N^{(m)})$ follows from item 2 of

Lemma 1, and Lemma 2. It is easy to see, that the optimal allocation of the customers to the servers in $N^{(m)}$, is an-m fold of the optimal allocation to $V^b(N)$. Also, the amount which is outsourced in $V^b(N^{(m)})$ is m times larger than the respective amount in problem $V^b(N)$. Thus, $V^b(N^{(m)}) = mV^b(N)$, implying that V^b is homogenuous of degree one.

■

Theorem 2 *For any $b \geq 0$, the cooperative game (N, V^b) is balanced.*

Proof: The proof follows directly from the sub-additivity and the homogeneity of degree 1 of the regular game $G = (N, V^b)$ (see Lemma 3) and Theorem 1. ■

Though by Theorem 2 we know that the core of the game is non-empty, we find it a challenging task to identify a core cost allocation. In the next two examples we let $(x_1, \ldots, x_n)$ represent the decision variable vector of a core cost allocation. The next example shows that it is possible that all core allocations contain negative entries. This is in contrast with our former paper [3], where we showed that under the type of cooperation defined there, core allocations with all its entries being non-negative always exist. Note that a negative entry means that a server, on top of his own customers being served by the grand coalition or being outsourced, is in fact being paid in order to join this coalition.

Example 7. Suppose that $b = 0$, and $n = 4$: $\mu_1 = 2000$, $\lambda_1 = 1000$, $\mu_2 = 1000$, $\lambda_2 = 900$, $\mu_3 = 800$, $\lambda_3 = 600$, $\mu_4 = 100$, and $\lambda_4 = 99$. Calculation shows that $V^b(N) = 6.11$ and the servers $\{1, 2, 3\}$ are open. Moreover, $V^b(N\backslash\{1\}) = 13.24$, which implies that $x_1 \geq -7.12$. $V^b(N\backslash\{2\}) = 2.84$, which implies that $x_2 \geq 3.27$. $V^b(N\backslash\{3\}) = 3.82$, which implies that $x_3 \geq 2.29$. $V^b(N\backslash\{4\}) = 5.42$, which implies that $x_4 \geq 0.69$. The three last inequalities imply that $x_2 + x_3 + x_4 \geq 6.26$. As $V^b(N)$ is smaller than the lower bound on $x_2 + x_3 + x_4$ we get that $x_1 \leq -0.15$. An example for a core allocation is $(x_1, x_2, x_3, x_4) = (-0.15, 3.27, 2.29, 0.69)$.

The next example shows a case where at least two servers must be paid by the others, and hence no core allocation, which is either non-negative, or consisting of a single negative entry, exists. Again, this is in contrast with the results about the model considered in our former paper [3] where we showed that if there exist core allocations with negative entries, then there exist core allocation with a single negative entry.

Example 8. Let $b = 0$, $n = 10$, $(\mu_1, \ldots, \mu_{10}) = (100, 78, 70, 65, 50, 45, 30, 20, 10, 5)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{10}) = (80, 60, 45, 20, 10, 20, 8, 12, 1, 4)$. $V^b(N) = 9.57$, which is attained when the first 8 servers are open. By similar calculations as in the previous example we get that $x_4 \geq -0.37$ and $x_5 \geq -0.48$. The lower bounds on all other 8 servers are positive, and when we sum them up we get that $x_1 + x_2 + x_3 + x_6 + x_7 + x_8 + x_9 + x_{10} \geq 9.90$. This means that $x_4 + x_5 \leq -0.32$. Moreover, the lower bound on x_1 is $x_1 \geq 3.09$. Considering the two coalitions $\{1, 4\}$ and $\{1, 5\}$ we get that $x_1 + x_4 \leq 3.02$ implying that $x_4 \leq -0.07$ and $x_1 + x_5 \leq 2.86$ implying that $x_5 \leq -0.23$. In particular, any core allocation comes with both servers 4 and 5 being paid by the others.

Remark 4. A somewhat related decision problem to the one considered here is that of equilibrium routing. Specifically, consider the above routing model, with a high outsourcing cost, so that no customer is outsourced, and the central planner of the system wants to minimize the overall mean waiting costs. In the equilibrium routing model, on the other hand, the customers act selfishly by deciding which queue to join in a way that minimizes their own waiting time. It is clear that customers are engaged in a non-cooperative game. This problem was dealt with in [6] who found the Nash equilibrium routing strategy. In general, it is not the same strategy as the socially optimal one defined above, yet, it shares some of its properties. For example, also in the equilibrium routing model only a subset of servers is utilized, a subset which is contained in the one utilized under the socially optimal routing. In [6] the equilibrium arrival rates are stated and it is now simple to find the coalitional costs under the equilibrium routing. We do not give any further details but claim that the resulting game is regular and homogeneous of degree one. Yet, it is not necessarily sub-additive so there is no guarantee for having a non-empty core. In fact, is it not hard to construct examples for such games which have an empty core.

4.2 Split of a total service capacity

Consider the same model as described in subsection 4.1, but with one key distinction: Now servers who cooperate share among themselves the total joint capacity while each server maintains her original arrival rate that she needs to serve. That means that the servers continue to work individually. Consider, for example, a communication network with n routers (servers) that route incoming calls (customers) to n different intermediate sites. Each incoming call is identified by the site it needs to reach. Each router routes calls to a single site. In such a communication network it is impossible to re-direct calls to other routers except the one that serves the required site. On the other hand, it is possible to re-allocate the routers capacities, as long as each router serves its own customers. Thus, in this second model of cooperation in queuing systems that we present here, customers of different lines require different service types, implying that customers of a certain line cannot switch to another line. The servers, on the other hand, are capable to perform all service tasks at their given capacity, so that each server can allocate its capacity among all lines. The servers of any coalition have also the option to rent some of their capacity.

To present our model formally, similarly to the one presented in subsection 4.1, we use the same notation: each server $i \in N = \{1, 2, \ldots, n\}$ is associated with its Poisson arrival rate λ_i and its exponential capacity μ_i, and $\mu(S) = \Sigma_{i \in S} \mu_i$ is the total service capacity rate of coalition $S \subseteq N$. The central controller of this system has to decide how much capacity to assign to each of the servers and how much capacity to rent in order to minimize the net cost which consists of the total steady-state congestion cost minus the revenue obtained by the capacity rental. As in subsection 4.1, we normalize the congestion cost to be one per unit, and the resulting normalized unit rate capacity rental price is b. We note that in this problem, except for the case where a server has no customers, i.e., its arrival rate is 0, all servers must be open. We let $W^b(S)$ denote the optimal net cost associated with coalition $S \subseteq N$. Next we provide an expression for the net cost of the grand coalition. A similar expression can be written for any coalition $S \subseteq N$.

Let $\Gamma \leq \mu(N)$ be the capacity that is used internally. Feasibility requires that $\Gamma > \lambda(N)$. Let $\Omega(N, \Gamma)$ be the optimal steady-state congestion cost for serving the customer arrival rates $\lambda_1, \ldots, \lambda_n$, by a total capacity of Γ. Let $\mu^I_{N,i}(\Gamma) > \lambda_i$ be the optimal capacity allocation to server $i \in N$ according to the cost function $\Omega(N, \Gamma)$. That means that $\sum_{i \in N} \mu^I_{N,i}(\Gamma) = \Gamma$. We consider the cooperative game $G = (N, W^b)$, where the characteristic function for the grand coalition (and similarly to any other coalition) is given by

$$W^b(N) = \min\{\Omega(N, \Gamma) - b(\mu(N) - \Gamma) : \ \lambda(N) < \Gamma \leq \mu(N)\}. \qquad (17)$$

Similarly to $V^b(N)$ in subsection 4.1, also $W^b(\emptyset) = 0$, and the set function $W^b(\cdot)$ is sub-additive, implying that any bargaining process will probably end up by forming the grand coalition. Problem $\Omega(N, \Gamma)$, for $\Gamma > \lambda(N)$, is formulated as follows:

$$\Omega(N, \Gamma) = \min \sum_{i=1}^{n} \frac{\lambda_i}{\mu^I_{N,i}(\Gamma) - \lambda_i}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} \mu^I_{N,i}(\Gamma) = \Gamma$$

$$\mu^I_{N,i}(\Gamma) > \lambda_i,\ 1 \le i \le n$$

It is well known, (see, e.g., [12], p. 331) that

$$\mu^I_{N,i}(\Gamma) = \lambda_i + \sqrt{\lambda_i}\ (\frac{\Gamma - \sum_{j\in N}\lambda_j}{\sum_{j\in N}\sqrt{\lambda_j}}), \quad i \in N \tag{18}$$

and hence

$$\Omega(N,\Gamma) = \frac{(\sum_{i\in N}\sqrt{\lambda_i})^2}{\Gamma - \sum_{i\in N}\lambda_j}. \tag{19}$$

Finally, the derivative of $\Omega(N,\Gamma)$ with respect to Γ equals

$$\frac{d\Omega(N,\Gamma)}{d\Gamma} = -\frac{(\sum_{i=1}^n\sqrt{\lambda_i})^2}{(\Gamma - \lambda(N))^2}. \tag{20}$$

Observe also that the game where the option of renting capacity is excluded, namely the game (N,Ω) where each coalition $S \subseteq N$ is associated with capacity $\mu(S)$, is a market game and therefore its core is non-empty. In particular, this game coincides with the game $G = (N, W^b)$ for b sufficiently small, i.e., when it is not desirable to rent any capacity, see Lemma 4.

For a general rental price b let $\mu_{N,i}$ be the optimal capacity allocated to server i for $1 \le i \le n$, and $\mu_{N,0}$ the optimal capacity rented according to $W^b(N)$. We also let $f_i(z_i) = \frac{\lambda_i}{z_i - \lambda_i}$ represent the steady-state congestion cost at server i, $1 \le i \le n$, as a function of the capacity $z_i > \lambda_i$ allocated to her, and $f_0(z_0) = -bz_0$ be the cost of renting z_0 units of capacity. Problem $W^b(N)$ can be written as follows:

$$W^b(N) = \min\{\sum_{i=0}^n f_i(z_i)\}$$

$$\text{s.t.} \quad \sum_{i=0}^n z_i = \mu(N)$$

$$z_i > \lambda_i,\ 1 \le i \le n$$

We next give an explicit expression for $W^b(N)$ defined in (17). For that sake, we write $W^b(N,\Theta)$ to show the dependence of $W^b(N)$ on the total capacity Θ. That means that $W^b(N) = W^b(N,\mu(N))$. Similarly to Lemma 2, we have here

Lemma 4

- *The function $W^b(N,\Theta)$ as a function of $\Theta > \lambda(N)$ is convex and continuously differentiable. Moreover, for any given value of $\Theta > \lambda(N)$ the solution of $W^b(N,\Theta)$ is unique.*

- *If b is too small, specifically if*

$$b \leq \frac{(\sum_{i=1}^{n} \sqrt{\lambda_i})^2}{(\mu(N) - \lambda(N))^2},$$

no capacity is rented. In this range of b, $W^b(N) = \Omega(N, \mu(N))$, and hence $W^b(N)$ is given by (19), where Γ is replaced by $\mu(N)$. Similarly, $\mu_{N,i}$ are given in (18), for $1 \leq i \leq n$, replacing again Γ by $\mu(N)$, and $\mu_{0,N} = 0$. Moreover, for any $i \in N$, $\frac{dW^b(N,\Theta)}{d\Theta}|_{\Theta=\mu(N)} = \frac{df_i}{dz_i}|_{z_i=\mu_{N,i}} = -\frac{(\sum_{i=1}^{n} \sqrt{\lambda_i})^2}{(\mu(N)-\lambda(N))^2} \leq -b$.

- *Otherwise, namely, if $b > \frac{(\sum_{i=1}^{n} \sqrt{\lambda_i})^2}{(\mu(N)-\lambda(N))^2}$,*

$$\mu_{N,i} = \lambda_i + \frac{\sqrt{\lambda_i}}{\sqrt{b}}, \quad 1 \leq i \leq n, \tag{21}$$

$\mu_{N,0} = (\mu(N) - \lambda(N)) - \sum_{i=1}^{n} \sqrt{\lambda_i}/\sqrt{b}$, and

$$W^b(N) = 2\sqrt{b} \sum_{i=1}^{n} \sqrt{\lambda_i} - b(\mu(N) - \lambda(N)). \tag{22}$$

Finally, for any $i \in N \cup \{0\}$, $\frac{dW^b(N,\Theta)}{d\Theta}|_{\Theta=\mu(N)} = \frac{df_i}{dz_i}|_{z_i=\mu_{N,i}} = -b$.

Remark 5. It is interesting to compare (15) with (21) and (16) with (22).

As discussed in subsection 4.1, the game $G = (N, W^b)$ is not a market game. Example 6 with $b = 0$ can be used to show that the game $G = (N, W^b)$ is not concave. By arguments similar to those used in the previous subsection, we conclude that

Theorem 3 *For any value for b, the game $G = (N, W^b)$ is regular, sub-additive and homogeneous of degree one. In particular, by Theorem 1, it is a balanced game.*

As for the game (N, V^b), also for the game (N, W^b), it is a challenging task to identify a core cost allocation.

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