REGULAR GAMES: CHARACTERIZATION AND TOTAL BALANCEDNESS OF REGULAR MARKET GAMES

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Regular games: characterization and total balancedness of regular market games

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Abstract

The conventional definition of a cooperative game $G(N, V)$ with a set of players $N = \{1, \ldots, n\}$ and a characteristic function V, is quite rigid to alterations of the set of players N. Moreover, it necessitates a large input of size that is exponential in n. We show that the characteristic function of many games allows a simple, efficient and flexible presentation of the game. We call such games *regular games.* In such games each player is characterized by a vector of quantitative properties, and the characteristic function value of a coalition depends only on the vectors of properties of its members. We show that some regular games in which players can cooperate with respect to some of their resources and whose immediate formulation does not fit the framework of market games, can nevertheless be transformed into market games and hence they are totally balanced.

1 Introduction

A general cooperative coalitional game is defined by a set of n players, $N = \{1, 2, \ldots, n\}$, that can break up into subsets. Any subset S of N, $\emptyset \subseteq S \subseteq N$, is called a *coalition*, where N itself is called the *grand-coalition.* Each coalition S is associated with a real non-negative value denoted by $V(S)$, where $V(\emptyset) = 0$. The value $V(S)$ is the total cost inflicted on the members of coalition S if they cooperate. The function $V : 2^N \to \Re$ is called a *characteristic function.* The pair $G = (N, V)$ is said to be a *cooperative game*

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with transferable utility. The total cost incurred by all players of N depends on the partition of N into disjoint coalitions, so that if N is partitioned into m disjoint coalitions, $S_1 \cup S_2 \cup \ldots \cup S_m = N$, $1 \leq m \leq n$, then the total cost is $\sum_{i=1}^{m} V(S_i)$. This conventional definition of a cooperative game has the advantage of being general and simple, i.e., any coalitional game can be casted into this framework. Yet, its main drawback is its input size. Specifically, in order to fully describe a game $G = (N, V)$, $2^n - 1$ are needed to be stored. The input size burden imposes a practical restriction on the size of games (number of players) that can be studied. Thus, the generality of this presentation comes at the cost of its limitations. In this paper we are going to discuss structured coalitional games, especially games in operations management and queueing systems, that allows a presentation by a much simpler and efficient form. Moreover, in many cases this presentation may lead one to realize that the game in totally balanced (namely, it and its subgames have non-empty cores).

We call the class of games that we focus on here *Regular Games*: In a regular game each player is characterized by a vector of quantitative properties, called *vector of properties* and the cost of a coalition of size m is a function V_m of the m vectors of properties of its members $m \geq 1$, but it is otherwise independent of the identity of the players. Moreover, the sequence of functions V_m for $m \geq 1$, satisfies a recursive relation. That means that all is needed in order to describe such a regular game $G = (N, V)$ is the permissible domain of vectors of properties, the vectors of properties of the players in N, and a function V_m that maps m vectors of properties into $\Re$, i.e., the input size to describe such a game is only of size $O(n)$. This presentation of regular games, has an additional important advantage, which is the possibility to easily extend their definition to any set of players once their vectors of properties are given. That means, that given a regular game $G = (N, V)$ where the players of N are associated with a set A of n vectors of properties, one can easily define infinitely many new compatible games by adding or removing vectors of properties to/from A.

In order to describe the rest of our results we review the main concepts in cooperative games that are relevant to this paper. The first question given a cooperative game is weather the grand-coalition is the socially optimal formation of coalitions. A sufficient condition for that is the sub-additivity of its characteristic function: A game $G = (N, V)$ is called *subadditive* if for any two disjoint coalitions S and T, $V(S \cup T) \leq V(S) + V(T)$. Subadditivity ensures that the socially best partition of the players of N to disjoint coalitions is when all players cooperate and join the grand-coalition. Subadditive games bear the concept of *economies of scope*, i.e., when each player, or set of players, contributes its own skills and resources, the total cost is no greater than the sum of the costs of the individual parts. On top of forming the grand-coalition, players of N need to establish a way that fairly allocate the cost $V(N)$ among themselves, so that no group of players may resist this cooperation and decide to act alone. Several fairness concepts have been proposed in the literature. One of the most appealing concepts is the *core* as it guarantees the stability of the grand-coalition: A vector $x \in \Re^n$ is said to be *efficient* if $\sum_{i=1}^{n} x_i = V(N)$, and it is said to be a *core cost allocation* of the game if it is efficient and if $\sum_{i \in S} x_i \leq V(S)$ for any $S \subseteq N$.

The collection of all core allocations, called the *core* of the game, is a simplex as it is defined by a set of linear constraints with n decision variables. Still, as the number of constraints that define the core is exponential in n, in fact it is $2^n - 1$, finding a core allocation for a given game may be, in general, an intricate task. Indeed, this issue coupled with the possibility that the core is empty, makes the problem of finding a core allocation a real challenge in some games. Moreover, even if we can prove the non-emptiness of the core, the question of finding a cost allocation in the core may be non-trivial.

A cooperative game $G = (N, V)$ is said to be *balanced* if its core is non-empty, and *totally balanced* if its core and the core of all its subgames are non-empty. Note subadditivity is a necessary condition for total balancedness: For any disjoint S and T for which $V(S) + V(T) < V(S \cup T)$, the subgame $(S \cup T, V)$ has an empty core since any efficient allocation of $V(S \cup T)$ among the players of $S \cup T$ will be objected by at least one of the coalitions S or T. The literature provides two conditions that are sufficient in order establish the total balancedness of a game.

- **Condition 1.** A game $G = (N, V)$ is a *concave game* if its characteristic function is concave, meaning that for any two coalitions $S, T \subseteq N$, $V(S \cup T) + V(S \cap T) \leq V(S) + V(T)$. Clearly, concave games are subadditive but not the other way around. It was shown in [13] that the core of a concave game possesses $n!$ extreme points, each of which being the vector of marginal contribution of the players to a different permutation of the players.

- **Condition 2.** A *market game*, see e.g., Chapter 13 in [11], is defined as follows: Suppose there are ℓ inputs. An *input vector* is a non-negative vector in $\Re^\ell_+$. Each of the n players possesses an initial commitment vector $w_i \in \Re^\ell_+$, $1 \leq i \leq n$, which states a nonnegative quantity for each input. Moreover, each player is associated with a continuous and convex cost function $f_i : \Re^{\ell}_{+} \rightarrow \Re_{+}$, $1 \leq i \leq n$. A profile $(z_i)_{i \in N}$ of input vectors for which $\sum_{i \in N} z_i = \sum_{i \in N} w_i$ is an *allocation.* The game is such that a coalition S of players looks for an optimal way to redistribute its members' commitments among its members in order to get a profile $(z_i)_{i \in S}$ of input vectors so as the sum of the costs across the members of S is minimized. Formally, for any $\emptyset \subseteq S \subseteq N$,

$$V(S) = \min \{ \sum_{i \in S} f_i(z_i) : z_i \in \Re^{\ell}_{+}, i \in S \text{ and } \sum_{i \in S} z_i = \sum_{i \in S} w_i \} \qquad (1)$$

Remark 1. In [11] it is assumed that the functions $f_i(z_i)$, $1 \leq i \leq n$, are non-increasing but as noted in [4] page 163 footnote 2, this in fact is not required.

Market games are not necessarily concave, but they are well-known to be totally balanced, see [12], Corollary 3.2.4. Unlike concave games whose core is fully characterized and has a closed form (see Condition 1), just a single core allocation based on competitive equilibrium prices, is known for general market games, see [11], p. 266. In fact, [14] proves that a game is a market game if and only if it is totally balanced. In particular, any concave game is a market game. However, if a game is not naturally formulated as a market game (see (1)), then the task of reformulating it as a market game (or showing that such a formulation does not exist), may be as intricate as proving directly that it is totally balanced (or that it is not). Thus, it seems that except for games that are either originally stated as market games, or are easily transformed to market games, this approach has its limits.

The two conditions stated above for total balancedness hold for general cooperative game $G = (N, V)$. Our objective in this paper is to provide a new condition for total balancedness of regular games. As we are going to show, regular games are not esoteric; in fact many well-known games are regular and therefore deepening our understanding on the total balancedness of such games is valuable and important.

In the next section we rigorously define regular games. In Section 3 we present two classes of regular games that we call *Aggregation Games*, and *General Resource Sharing Games.* In Section 4 we focus on a subclass of general resource sharing games called *Regular Market Games* that can be transformed into market games, see (1), and therefore they are totally balanced and a specific core cost allocation based on competititve equilibrium prices can be derived. In Section 5 we present and study two regular queueing games that deal with servers' cooperation and show that they are regular market games and therefore they are easily proved to be totally balanced.

2 Regular games

Consider a cooperative game $G = (N, V)$ that is defined by its set of players N and its characteristic function V that satisfies the following conditions: Each player $i \in N$ is fully characterized by the quantities of a given number $\kappa \geq 1$ of resources that he/she owns. Let number the resources that are considered by $\ell = 1, \ldots, \kappa$, so that player $i \in N$ is associated with a vector $y^i \in \Re^\kappa$, called a *vector of properties*, and y^i_ℓ specifies the quantity of resource ℓ, $1 \leq \ell \leq \kappa$, owned by player $i \in N$. Moreover, the characteristic function value of coalition $S \subseteq N$, namely $V(S)$, is a function only of the $|S|$ vectors of properties that characterize the members of S and not of the players' identities. Some of the κ resources can be shared among the members of a coalition, where the other resources are individual resources that are not sharable. The characteristic function $V(S)$ of any coalition $S \subseteq N$, is the cost induced by the members of S when the sharable resources are used by coalition S according to the rules of the game. In some games all resources are sharable, and in other games only some resources are sharable and the others are serving as characteristics (parameters) of the players. As an example of these two types of resources consider a set of service providers whose stream of customers is given, i.e., customers must get service from the service provider that they choose, but the service providers can share their service capacities in the sake of minimizing the overall congestion of the system. In this example, each service provider is assigned a vector of properties of size 2 for its service rate and its arrival rate. The capacity (service rate) is a sharable property where the arrival rate is non-sharable.

As we are going to show, under certain conditions it is natural to extend the definition of such games and their characteristic function to any set of players, not necessarily those who are physically involved in the particular game $G = (N, V)$. Note that this extension is both in terms of different set of players and in the number of players. I.e., the characteristic function in such games can be applicable to any collection of vectors of properties. We call such games *Regular Games*. Apparently, the class of regular games is quite large and it contains many well-known and interesting games. In this section we will formalize the notion of *Regular Games* and propose an alternative definition for such games. We start by presenting a well-known game, the airport game, see [9] and [10], which is later shown to satisfy the conditions of a regular game:

Example 1 An airport with a single runway serves m different types of aircrafts. An aircraft of type k, $1 \leq k \leq m$, is associated with a cost $c_k \geq 0$ of building a runway to accommodate aircrafts of its type. Let N_k be the set of aircraft of type k landing in a day, and $N = \cup_{k=1}^{m} N_k$. The characteristic function for any $S \subseteq N$ is given by $V(S) = \max\{c_k | S \cap N_k \neq \emptyset,\ 1 \leq k \leq m\}$ and $V(\emptyset) = 0$. This conventional presentation of the airport game reveals its limitations as it is defined just for the given m aircraft types, and, moreover, for each given type k, $1 \leq k \leq m$, it is defined just for the landings in the set N_k. Each change, as adding a landing of a new type, or even adding a new landing of one of the given types, necessitates the definition of a new game. However, it is easy to see that this definition for the game can be easily generalized to any set of players each of which having its parameter c_k. This will formally be done below.

In a regular game each potential player j is associated with a vector of properties y^j of size $\kappa \geq 1$, that may be required to satisfy some feasibility constraints of the form $y^j \in D$, where $D \subseteq \Re^\kappa$. Let $y^{(m)}$ denote a sequence of m vectors of properties $y^1, \ldots, y^m$ in D. The following two definitions formally define a regular game:

Definition 1 *An infinite sequence of symmetric functions $V_0, V_1, \ldots, V_m, \ldots$ is said to be Infinite Increasing Input-Size Symmetric Sequence (IIISSS) of functions for given integer $\kappa \geq 1$, and a subset D of $\Re^\kappa$, if*

- $V_0 \equiv 0$;
- *For any $m \geq 1$, $V_m : D^m \to \Re$;*
- *There exists a vector $y^0 \in D$ such that $V_1(y^0) = 0$ and for any given sequence of $m-1$ vectors of properties $y^{(m-1)} = (y^1, \ldots, y^{m-1}) \in D^{m-1}$, $V_{m-1}(y^{(m-1)}) = V_m(y^{(m-1)}, y^0)$.*

For a given IIISSS of functions $(V_m)_{m \geq 0}$, V_m receives as input m vectors of size κ, each is a member of the set D, and it returns a real value. As the functions V_m are symmetric, the order of the m input vectors has no affect on the value of the function. In other words, let $\phi(y^1, \ldots, y^m)$ be any permutation of $(y^1, \ldots, y^m) \in D^m$. The symmetric property of the function V_m implies that $V_m(y^1, \ldots, y^m) = V_m(\phi(y^1, \ldots, y^m))$. The third item of the definition guarantees that the definition of the various functions of the

IIISSS of functions is consistent, i.e., it excludes the possibility that there exist two functions V_ℓ and V_k for $\ell \neq k$, $\ell, k \geq 1$, where each is defined by a different mathematical expression. This is achieved by requiring to have a *null vector of properties* $y^0 \in D$ that connects the different functions through a forward recursion. For example, suppose that each player i is associated with a certain real number α_i that represents its liability and that the value of a coalition is the average liability of its members. In such a case let $\kappa = 2$, player i is associated with a vector $y^i = (\alpha_i, 1)$, the null vector is $y^0 = (0,0)$ and $D = \{(0,0)\} \cup \{(x,1) : x \in \Re\}$. Given m vectors of properties $y^{(m)} \in D^m$, $y^i = (\alpha_i, \beta_i) \in y^{(m)}$, $i = 1, \ldots, m$, the value $V_m(y^{(m)}) = \sum_{i=1}^m \alpha_i / \sum_{i=1}^m \beta_i$, i.e., $V_m(y^{(m)})$ is the average of the liabilities of the non-null vectors in D. Note that the choice of y^0 as the zero-vector is a quite natural choice for a null vector that holds in many other games. But in some games y^0 is not necessarily the zero vector. Consider a similar example to the above one with a characteristic function that returns for any coalition the product of the liabilities in the coalition divided by the number of players in the coalition, i.e., $V_m(y^{(m)}) = \Pi_{i=1}^m \alpha_i / \sum_{i=1}^m \beta_i$. In such a case the null vector $y^0 = (1,0)$, and $V_1(y^0)$ is defined as 0.

Definition 2 *A game $G = (N, V)$ is called regular if there exists a set $D \in \Re^\kappa$, such that player i, $i \in N$, is associated with a vector of properties $y^i \in D$, and there exists an IIISSS of functions $V_\ell : D^\ell \to \Re$, $\ell \geq 0$, such that for any $S \subseteq N$, $V(S) = V_{|S|}(y^i|_{i \in S})$.*

Observation 1 *A market games $G = (N, V)$, as described in Condition 2 in Section 1, is not a regular game in general, as the cost function of a player may depend on its identity. That means that two players having exactly the same endowments and playing as singletons may be associated with different costs due to different cost functions. A market game is a regular game if all individual cost functions $f_i(x)$, $i \in N$, are identical. Later we will get back to such games, to be called regular market games.*

In the next section we identify two types of IIISSS of functions that generate many well-known regular games.

3 Two classes of regular games

3.1 Aggregation games

In an *aggregation game*, all resources are sharable, i.e., individual properties do not exist. Moreover, when two players, say i and j, cooperate, their combine effect on the cost of the coalition they join, is as if they were replaced by a single new player having a vector of properties $g(y^i, y^j)$, for some symmetric function $g : D^2 \to D$ that satisfies the commutative and the associative laws. That means that $g(y^i, y^j) = g(y^j, y^i)$, for $i \neq j$, and that $g(y^i, y^j), y^k) = g(y^i, g(y^j, y^k))$ for $k \notin \{i, j\}$. Such a function g is called an *aggregation function*. For simplicity denote the aggregation function of m vectors of properties by $g^{m-1} : D^m \to D$, i.e., $g^{m-1}(y^1, \ldots, y^m) \in D$. A given cost function for one player, namely V_1, together with an aggregation function $g : D^2 \to D$, generate a corresponding IIISSS of functions in the following way: $V_m(y^1, \ldots, y^m) = V_1(g^{m-1}(y^1, \ldots, y^m))$. Next we present two specific aggregation games:

Contd. of Example 1: In the airport game $\kappa = 1$, $D = \Re_0^+$, $g : D^2 \to \Re$, where the aggregation function $g(c_1, c_2) = \max(c_1, c_2)$ for any $(c_1, c_2) \in D^2$, and $g^{m-1}(c_1, \ldots, c_m) = \max(c_1, \ldots, c_m)$ for any $(c_1, \ldots, c_m) \in D^m$. For any $c \geq 0$, $V_1(c) = c$. Thus, $V_m(c_1, \ldots, c_m) = V_1(\max(c_1, \ldots, c_m)) = \max(c_1, \ldots, c_m)$.

Example 2 In [1] we dealt with what seems to be the simplest (but most revealing) possible model of cooperation in service systems. This model is based on the assumption that when a set of servers cooperate, they work as a single server whose service rate is the sum of the individual service rates. Moreover, this combined server serves their joint stream of arrivals. More precisely, let $N = \{1, \ldots, n\}$ be a set of n $M/M/1$ queueing systems. They can cooperate in order to minimize the steady-state congestion in the combined system. Queueing system i is associated with its own exponential service rate μ_i and its own Poisson arrival rate of customers λ_i, $\lambda_i < \mu_i$, $i \in N$. Cooperation of a set $S \subseteq N$ in this model results in a single $M/M/1$ queue whose capacity is $\mu(S) = \sum_{i \in S} \mu_i$, and whose arrival rate $\lambda(S) = \sum_{i \in S} \lambda_i$. For any coalition $S \subseteq N$ the congestion of S is given by

$$V(S) = \frac{\lambda(S)}{\mu(S) - \lambda(S)}.$$

Next we present this game as a regular aggregation game: each player, namely each queueing system, is associated with a vector of properties of size $\kappa = 2$, $y^0 = (0, 0)$, and $D = \{0, 0\} \cup \{(\lambda, \mu) | 0 \leq \lambda < \mu\} \subset (\Re^+)^2$. Let $V_1(y^0) = 0$, and for $(\lambda, \mu) \in D \backslash \{0, 0)\}$, $V_1(\lambda, \mu) = \frac{\lambda}{\mu - \lambda}$. The aggregation function g that combines two vectors of properties in D into one is $g((\lambda_1, \mu_1), (\lambda_2, \mu_2)) = (\lambda_1 + \lambda_2, \mu_1 + \mu_2)$, thus
$g^{m-1}((\lambda_1, \mu_1), \ldots, (\lambda_m, \mu_m)) = g(\sum_{i=1}^m \lambda_i, \sum_{i=1}^m \mu_i)$. Let $V_m(y^1, \ldots, y^m) = V_1(g^{m-1}((\lambda_1, \mu_1), \ldots, (\lambda_m, \mu_m)))$. Thus, if $g^{m-1}((\lambda_1, \mu_1), \ldots, (\lambda_m, \mu_m)) \neq$

y^0, then $V_m(y^1, \ldots, y^m) = \sum_{i=1}^m \lambda_i / \sum_{i=1}^m (\mu_i - \lambda_i)$, and otherwise $V_m(y^1, \ldots, y^m) = 0$.

It was shown in [1] that this game, which is neither concave nor it possesses the shape of a market game (see (1)), is totally balanced. In fact, the non-negative part of the core has been fully characterized in [1]. In particular, $x_i = \frac{\lambda_i}{\lambda(N)} V(N)$, $i \in N$, is a core allocation.

3.2 General resource sharing games

Next we define a *general resource sharing game.* In such games players keep their individuality, i.e., players also have individual properties that cannot be shared with their mates in a coalition. This is in contrast to aggregation games described above, where players were amalgamated into one "big" player that used all their properties as given by their vector of properties. In general resource sharing games, some properties owned by the players cannot be shared with other members in the coalition that he/she joins, while the other properties are shared in an additive way that optimizes the cost of the coalition. The set $SP = \{1, \ldots, s\}$ for some $s \leq \kappa$ is the set of properties that players do share, while the set $IP = \{s+1, \ldots, \kappa\}$ is the set of individual properties that players do not share. The set D is assumed to be convex in the shared properties, and it consists of vectors in $\Re^\kappa$ whose first s entries are non-negative. In particular, the null vector $y^0 \in D$ is the zero vector in $\Re^\kappa$. For any vector of properties $y \in D$, let the set of vectors of properties $Y(y) = \{y' \in D | y'_\ell = y_\ell \text{ for } \ell \in IP\}$. That means that the set $Y(y)$ contains all possible vectors of properties in D that their individual properties, i.e., the entries in IP, are identical to those of y. Let the set $E = \{y \in \Re^\kappa : y_\ell \geq 0,\ \ell \in SP,\ and\ y_\ell = 0 \ \ for\ \ell \in IP\}$. In addition, let $f : D \to \Re$, and $h : E \to \Re$, be two continuously differentiable functions such that for any $y \in D$

$$V_1(y) = \min\{f(x) + h(y - x) | x \leq y, \text{ and } x \in Y(y)\},\ y \in D \qquad (2)$$

with $f(\vec{0}) = h(\vec{0}) = 0$, where $\vec{0} = y^0 \in \Re^\kappa$ is the zero-vector. Equation (2) clearly implies that $y - x \in E$. Note that (2) is well defined as the minimization is over a closed set where the first s entries of the vector x are non-negative and they are bounded from above by the respective entries of the vector y. We call the function h the *extra function.* It is possible that the extra function is the null function. In such a case, $V_1(y) = f(y)$ for any $y \in D$, and the game is called a *simple general resource sharing game.*

Next we define an IIISS of functions that is associated with given continuously differentiable functions f and h based on (2), leading to the existence of an associated cooperative regular game:

Claim 1 *1. Given an integer $\kappa \geq 1$, a convex set $D \subset \Re^\kappa$, $y^0 = \vec{0}, y^1, \ldots, y^m \in \Re^\kappa \cap D$ and continuously differentiable functions f and h such that (2) holds with $V_1(y^0) = f(y^0) = h(y^0) = 0$, then a unique IIISSS of functions exists so that: $V_0 \equiv 0$, and for $m \geq 2$*

$$V_m(y^1, \ldots, y^m) = \min\{f(\tilde{y}^m) + V_{m-1}(x^1, \ldots, x^{m-1}) : \\ x^i \in Y(y^i) \text{ for } i = 1, \ldots, m, \text{ and } \tilde{y}^m + \sum_{i=1}^{m-1} x^i = \sum_{i=1}^{m} y^i\} \quad (3)$$

2. If, in addition, $h \equiv 0$, i.e., the extra function is the null function, then Equation (3) boils down to

$$V_m(y^1, \ldots, y^m) = \min\{\sum_{i=1}^{m} f(\tilde{y}^i) : \\ \tilde{y}^i \in Y(y^i) \text{ for } i = 1, \ldots, m, \text{ and } \sum_{i=1}^{m} \tilde{y}^i = \sum_{i=1}^{m} y^i\} \quad (4)$$

Proof:

We prove below the first item. The proof for the second item is similar are hence omitted. Given (3), the only property that we need to show is that the functions V_m, $m \geq 2$, are symmetric. In simple general resource sharing games where the extra function is the null function (see (4)), symmetry is trivial. Otherwise, (3) implies that

$$V_2(y^1, y^2) = \min\{f(\tilde{y}^2) + V_1(y^1 + y^2 - \tilde{y}^2) : \\ \tilde{y}^2 \in Y(y^2) \text{ and } y^1 + y^2 - \tilde{y}^2 \in Y(y^1)\}. \quad (5)$$

By (2), we get that

$$V_1(y^1 + y^2 - \tilde{y}^2) = \min\{f(x^1) + h(y^1 + y^2 - \tilde{y}^2 - x^1) : \\ x^1 \in Y(y^1 + y^2 - \tilde{y}^2) \text{ and } y^1 + y^2 - \tilde{y}^2 - x^1 \in E\}. \quad (6)$$

Note that $Y(y^1 + y^2 - \tilde{y}^2) = Y(y^1)$, as the IP entries of y^2 and $\tilde{y}^2$ are identical. Thus,

$$V_2(y^1, y^2) = \min\{f(\tilde{y}^2) + f(x^1) + h(y^1 + y^2 - \tilde{y}^2 - x^1) : \\ x^1 \in Y(y^1),\ \tilde{y}^2 \in Y(y^2) \text{ and } y^1 + y^2 - \tilde{y}^2 - x^1 \in E\}, \quad (7)$$

proving the symmetry for $m = 2$. The same arguments can be reapplied to prove that V_m is symmetric for any $m > 2$. ■

Observation 2 As argued in Observation 1, market games, as defined in Condition 2 of Section 1, are not necessarily regular games. However, market games where all potential players share the same cost function, are regular games. In particular, simple general resource sharing games (see (4)), where the corresponding function f is convex in the shared properties, are also market games. This is because the cost function associated with a player whose vector of properties is $y \in D$ is $V_1(y) = f(y)$, where f is convex. This implies that simple general resource sharing games that are defined by a convex function f are totally balanced.

Example 3 This example considers loss systems. The parameters are as defined in Example 2 with one key distinction: There are no waiting places. Thus, a customer who finds the server busy is lost for good. It is well known that the loss probability in case of an arrival rate of λ and service rate of μ is $\lambda/(\mu + \lambda)$. Suppose there exists a set N of n servers. The arrival and service rates of server i are λ_i and μ_i, respectively, $1 \leq i \leq n$. Servers can cooperate so as to minimize their total loss rate among themselves. Assume that the servers in a coalition cannot redirect customers but they can reallocate their total service capacity so as to minimize the loss rate. In terms of a cooperative game, let $S \subseteq N$ be a coalition of servers and let $W(S)$ be the rate of lost customers among the customers that arrive to the servers of S when they cooperate. By that we mean how many customers per unit of time among the potential arrival rate of $\Sigma_{i \in S} \lambda_i$ find their server busy upon arrival and hence are lost for good. The game (N, W) is associated with the following characteristic function:

$$W(S) = \min\{\sum_{i \in S} \frac{\lambda_i^2}{\lambda_i + c_i} | \; c_i \geq 0, i \in S\}$$

$$\text{s.t.} \qquad \sum_{i \in S} c_i = \sum_{i \in S} \mu_i$$[1]

[1]It is not hard to see that the optimal c_i, $i \in N$, are proportional to the arrival rates, making all servers being busy with equal probabilities. In particular,

$$W(S) = \frac{(\sum_{i \in S} \lambda_i)^2}{\sum_{i \in S} \lambda_i + \sum_{i \in S} \mu_i}, \; \subseteq S \subseteq N.$$

It is easy to see that this game fits the presentation of a market game, and therefore a core cost allocation based on competitive equilibrium prices can be derived, see Section 1.

Next we present this game as a simple general resource sharing game. For that sake note that the service rate corresponds to the shared property while the arrival rate to the individual property. I.e, $\kappa = 2$, $s = 1$, $SP = \{1\}$, $IP = \{2\}$, $y^0 = (0,0)$, and $D = \{(\mu,\lambda) : \lambda,\mu \geq 0\}$. The cost of any single player that is associated with a vector of properties $(\mu,\lambda) \in D$, is $V_1((\mu,\lambda)) = f(\mu,\lambda) = \frac{\lambda^2}{\mu+\lambda}$, thus, by using Claim 1, second item, for $m \geq 2$ and $(\mu_i,\lambda_i) \in D$ for $i = 1,\ldots,m$, we have (see (3))

$$V_m((\mu_i,\lambda_i)_{i=1,\ldots,m}) = \min_{c_1,\ldots,c_m} \{\sum_{i=1}^{m} \frac{\lambda_i^2}{\lambda_i + c_i} : c_i \geq 0,\ 1 \leq i \leq m \text{ and } \sum_{i=1}^{m} c_i = \sum_{i=1}^{m} \mu_i\}.$$

Note that in this simple resource sharing game, the function f is convex with respect to the shared property, and therefore the game is totally balanced since it is a market game.

Example 3 is a particular case where the game is both a regular and a market game. In the next section we focus on such games, which we call *regular market games.* Clearly, simple general resource sharing games with f convex in the shared properties, as in Example 3, fall within this class.

4 Regular market games

In this section we identify further conditions that if satisfied by a general resource sharing game, then the game is guaranteed to be a regular market game and as such it is totally balanced. In the next theorem we show that a general resource sharing game, see (2), that is not necessarily simple, but its f function is separable and convex in the shared properties, and its extra h function is linear, can be transformed into a regular market game, and as such it is totally balanced. In particular, a general resource sharing game with a single shared property, where the function f is convex in this property, and the extra function h is linear, is totally balanced. The proof of the theorem is constructive in the sense that it presents an algorithm that transforms such a general resource sharing game into the form of a regular market game.

Theorem 1 *A general resource sharing game, which is defined by a function f that is convex and separable in the shared properties and by a linear extra* *function h, (see (2)), can be transformed into a regular market game, and therefore it is totally balanced.*

Proof: We show below that a general resource sharing game that satisfies the conditions of the theorem can be transformed into a simple general resource sharing game with a function $\phi : \Re^\kappa \to \Re$ that is convex in the shared properties, such that $V_1(y) = \phi(y)$ for any $y \in D$, $\phi(y^0) = 0$, and functions V_m for $m > 1$ that are defined as in (4). The proof will be concluded by applying Observation 2.

First, throughout the rest of the proof, we fix the individual properties at a certain level $(y'_{s+1}, \ldots, y'_\kappa)$ and we consider them as given parameters. As the function f is separable in the shared properties, we can rewrite it as a sum of s functions, one for each of the shared properties, $f_i : \Re^+ \to \Re$, for $i \in SP$, where $f(x_1, \ldots, x_s, y'_{s+1}, \ldots, y'_\kappa) = \sum_{i=1}^s f_i(x_i)$. We note that the functions f_i for $i \in SP$, may depend on the individual properties $(y'_{s+1}, \ldots, y'_\kappa)$. The extra function h is linear in the shared properties, and as $h(\vec{0}) = 0$, there exist s coefficients $b_1, \ldots, b_s$, such that for any $x \in E = \{x \in \Re^\kappa : x_\ell \geq 0,\ \ell \in SP, \text{ and } x_\ell = 0 \text{ for } \ell \in IP\}$, $h(x) = \sum_{i=1}^s b_i x_i$. The problem associated with (2) for $y \in D$ whose individual properties are equal to $(y'_{s+1}, \ldots, y'_\kappa)$ can be rewritten in this case as:

$$V_1(y) = \min\{\sum_{i=1}^s f_i(x_i) + \sum_{i=1}^s b_i(y_i - x_i)|\ \ 0 \leq x_i \leq y_i \text{ for } i \in SP\} \qquad (8)$$

For any given $y \in \Re^\kappa$ with individual properties fixed at $(y'_{s+1}, \ldots, y'_\kappa)$ this optimization problem consists of s variables $x_1, \ldots, x_s$, and $2s$ linear constraints of the form $x_i \leq y_i$ and $x_i \geq 0$ for $i \in SP$. The KKT conditions, with λ_i being the lagrange multiplier of the constraint $x_i \leq y_i$ for $i \in SP$, are necessary and sufficient as the objective function is convex and the constraints are linear. Thus, any solution to $V_1(y)$, (see (8)), satisfies the following $6s$ conditions:

- $\frac{\partial f_i}{\partial x_i} - b_i + \lambda_i \geq 0, \quad i \in SP;$
- $x_i(\frac{\partial f_i}{\partial x_i} - b_i + \lambda_i) = 0, \quad i \in SP;$
- $x_i \geq 0, \quad i \in SP;$
- $x_i \leq y_i, \quad i \in SP;$
- $\lambda_i(y_i - x_i) = 0, \quad i \in SP;$

- $\lambda_i \geq 0, \quad i \in SP$;

The solution of the KKT conditions results in values of the $2s$ variables, which we denote by $x_1^*(y), \ldots, x_s^*(y)$ and $\lambda_1^*(y), \ldots, \lambda_s^*(y)$. These conditions imply that for any $y \in \Re^\kappa$, with individual properties fixed at $(y'_{s+1}, \ldots, y_\kappa)$, and $i \in SP$, if $y_i > 0$ then either (i) $x_i^*(y) = 0$ implying that $\frac{\partial f}{\partial x_i}|_{x_i=0} \geq b_i$; (ii) $0 < x_i^*(y) < y_i$ implying that $\frac{\partial f}{\partial x_i(y)}|_{x_i=x_i^*(y)} = b_i$; or (iii) $x_i^*(y) = y_i$ implying that $\frac{\partial f}{\partial x_i}|_{x_i(y)=y_i} \leq b_i$.

The transformation of a general resource sharing game into a simple general resource sharing game is done by applying the following procedure for each $i \in SP$, for the given values of the individual properties $(y'_{s+1}, \ldots, y'_\kappa)$:

1. If $\frac{df_i}{dx_i}|_{x_i=0} \geq b_i$, then let $x_i^*(y) = 0$ and $\phi_i(y_i) = by_i$.

2. If there exists a finite solution $x_i^*(y) > 0$ to the equation $\frac{df_i}{dx_i}|_{x_i=x_i^*(y)} = b_i$, then let $\phi_i(y_i) = f_i(\min\{y_i, x_i^*(y)\}) + b_i \max\{y_i - x_i^*(y), 0\}$.

3. Otherwise, i.e., if $\frac{df_i}{dx_i}|_{x_i=x_i^*(y)} < b_i$ for any $x_i^*(y) > 0$ then let $\phi_i(y_i) = f_i(y_i)$.

Let $\phi(y) = \sum_{i=1}^s \phi_i(y_i)$. In fact, $\phi(y)$ represents the cost of any player with a vector of properties $y \in D$ whose individual properties are set at $(y'_{s+1}, \ldots, y'_\kappa)$. The functions ϕ_i for $i \in SP$ are convex in their argument, and therefore, as a sum of convex functions, $V_1(y) = \phi(y)$ is convex in the shared properties, resulting in a simple general resource sharing game with a convex function ϕ that defines V_1. According to Observation 2, this is a regular market game. ■

Example 4 Assume the same setting as in Example 3 but now there is a cost α_i per customer lost at server i, and in addition, there is an option to rent out some of the capacity of the coalition at a revenue of r per unit rate of capacity. Thus, each player is associated with the shared property $\mu_i > 0$, and the individual properties $\lambda_i > 0$, and $\alpha_i > 0$, implying that $\kappa = 3$, $s = 1$, and $y^0 = (0, 0, 0)$ is the null vector. Also, $D = \{(\mu, \lambda, \alpha) : \lambda \geq 0, \mu \geq 0\}$. For $y = (\mu_i, \lambda_i, \alpha_i) \in D$, $y \neq y^0$, let

$$V_1(y) = \min_{c_i}\{\frac{\alpha_i \lambda_i^2}{c_i + \lambda_i} - r(\mu_i - c_i) : 0 \leq c_i \leq \mu_i\}.$$

This game is a general resource sharing game as for any vector $(\mu_i, \lambda_i, \alpha_i) \in D \backslash \{y^0\}$ define $f((\mu_i, \lambda_i, \alpha_i)) = \frac{\alpha_i \lambda_i^2}{c_i + \lambda_i} - r\mu_i$ and the extra function $h(x, 0, 0) =$

rx. As f is convex in the shared property, and the extra function is linear in the shared property, the resulting game can be reduced into a regular market game (see Theorem 1), and therefore the game is totally balanced. In fact a core cost allocation based on equilibrium competitive prices can be calculated for this game once it is casted into the form of a regular market game.

Future research should try to identify other regular games that can be proved to be regular market games as such games are known to be totally balanced and a specific core allocation can be computed via competitive equilibrium prices. Thus, any such results can shade light on the balancedness of cooperative games. In the next section we present two examples of cooperative games in queueing systems that we transform into regular market games, and based on that we compute a core cost allocation for each of them.

5 Applications in line balancing

Line balancing and resource pooling in service operations are an important practice. These two concepts are widely used for achieving a competitive advantage of a firm over its rivals. Some papers have considered resource pooling in the context of cooperative games, see e.g. [1], [6], [7], [15], [16] and [17]. In this section we present two line balancing models, where the basic system consists of a number of $M/M/1$ queueing systems. The first model redistributes the arrival rates while holding the original servers' capacities intact, and in the second, the capacities are redistributed, while the original arrival rates are preserved. In both cases the option of outsourcing/renting out exists. The question is always how to fairly allocate the total cost among the servers.

5.1 The unobservable routing game with outsourcing

Consider a system that provides one kind of service in a certain facility by a number of servers, where each is associated with its own capacity, i.e., some servers may be faster than others. In this subsection the customers cannot choose their server, though they do reach the facility due to a specific server they wish to be served with. This description, for example, fits clinics in remote locations in undeveloped countries that specialize in one kind of treatment as vision correction surgeries; the servers are ophthalmologists, and the customers are their clients. The clients are usually pre-examined by one of the specialists at their village. Upon arrival, a central controller routes the customers among the different servers. The central controller has also the option to outsource some customers to another facility, which provides the same type of service, in order to reduce the congestion, but this comes at a cost. The objective is minimizing the sum of the congestion cost at the facility plus the outsourcing cost.

Let $N = \{1, 2, \ldots, n\}$ be a set of servers. Server $i \in N$ has the capacity to serve customers in an exponentially distributed service time with mean $\mu_i^{-1} > 0$, and it faces a stream of Poisson arrivals at a rate λ_i, $\lambda_i \geq 0$. A central controller reroutes the total arrival rate $\lambda(N) = \sum_{i=1}^{n} \lambda_i$ among the servers of N, with the option of outsourcing some of it at a constant cost per unit rate outsourced. Congestion is measured by the mean steady-state number of customers in the system under the optimal split of arrivals to the servers. For simplicity, assume that the cost per unit of congestion is one, and accordingly the cost per unit rate outsourced is $b > 0$. Define a cooperative game $G = (N, V^b)$ where each coalition $S \subseteq N$ of servers is associated with a cost $V^b(S)$ that represents the minimum cost over all possible routings for the arrival rate $\lambda(S)$ among the servers of S and the external service provider that charges b per unit of arrival rate that is outsourced.

Lemma 1 *The unobservable routing game with outsourcing is a regular market game, thus it is totally balanced for any given set of players and any value of b.*

Proof: This game can be formulated as a regular market game, see Section 4, with a single shared property, which is the arrival rate, and a single individual property, which is the service rate: Let $D = \{(\lambda, \mu) : \lambda \geq 0,\ \mu > 0\} \cup \{(0, 0)\}$. Let $f((z, \mu)) = \frac{z}{\mu - z}$ if $z < \mu$, and if $z \geq \mu$, let $f((z, \mu)) = \infty$. $f((z, \mu))$ represents the congestion cost at a server with an arrival rate z and service rate μ. It is easy to see that f is convex in z. Let the extra function $h((x, 0)) = bx$ represent the outsourcing cost of x customers per unit time. Thus, $V_1((\lambda, \mu)) = \min\{f((z, \mu)) + h((\lambda - z, 0)) :\ 0 \leq z \leq \lambda\}$ and $V_n((\lambda_i, \mu_i)|_{i=1\ldots n})$ for any $n \geq 2$ is defined as in Equation (3). Thus, this general resource sharing game is in particular a regular market game, see Theorem 1, and therefore it is totally balanced. ∎

Lemma 1 implies, among other properties, that the game $G = (N, V^b)$ is a market game and therefore it possesses a core allocation which is based on competitive equilibrium prices (see Section 1). In order to compute this core cost allocation we first need to present the game as a market game (see Condition 2 in Section 1) as currently it is not presented in such a form. This transformation is done by following the results of Section 4.

In the unobservable routing game with outsourcing each player $i \in N$ is associated with a vector of two properties, where the arrival rate λ_i is the shared property, and where the capacity rate μ_i is the individual property. Following the proof of Lemma 1, we define the function $\phi(\lambda, \mu)$:

$$\phi(\lambda, \mu) = V_1(\lambda, \mu) = \min\{\frac{z}{\mu - z} + b(\lambda - z)| \ \ 0 \leq z \leq \lambda\}.$$

In order to derive a closed form expression for $\phi(\lambda, \mu)$, let $\mu^b = b^{-1}$. Any server with a capacity rate $\mu \leq \mu^b$ will be closed at optimality, since $\frac{d(\frac{z}{\mu - z})}{dz}|_{z=0^+} = \frac{1}{\mu} \geq b$, meaning that it is cheaper to outsource customers then serving them by this server (even under the ideal case of not having to wait). For such servers let $z^b(\mu) = 0$, and $\phi(\lambda, \mu) = b\lambda$. Otherwise, namely when $\mu > \mu^b$, $z^b(\mu)$ is the maximum rate of customers that this server serves before using the outsourcing option. In general,

$$z^b(\mu) = \max\{0, \mu - \sqrt{\mu/b}\}. \tag{9}$$

In fact, $z^b(\mu)$ is the solution of the equation $\frac{d(\frac{z}{\mu - z})}{dz} = b$ for $\mu > \mu^b$. We note that $z^b(\mu)$ is a non-decreasing function of b. Therefore, for any $\mu > 0$,

$$\phi(\lambda, \mu) = \begin{cases} \frac{\lambda}{\mu - \lambda} & \text{if } \lambda \leq z^b(\mu) \\ \frac{z^b(\mu)}{\mu - z^b(\mu)} + b(\lambda - z^b(\mu)) & \text{otherwise.} \end{cases}$$

As importantly,

$$V^b(N) = \min\{\sum_{i=1}^{n} \phi(z_i, \mu_i) : \sum_{i=1}^{n} z_i = \lambda(N) \text{ and } z_i \geq 0 \text{ for } i = 1, \ldots, n\}. \tag{10}$$

A presentation as a market game, see Condition 2 in Section 1, can be obtained by indexing the servers in a non-increasing order of their capacities (this can be done without loss of generality), i.e., $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n$, and defining the convex function $g_i(z) = \phi(z, \mu_i)$. Thus,

$$V^b(N) = \min\{\sum_{i=1}^{n} g_i(z_i) : \sum_{i=1}^{n} z_i = \lambda(N) \text{ and } z_i \geq 0 \text{ for } i = 1, \ldots, n\}. \tag{11}$$

Consider the set of servers N: servers with a capacity rate of $\mu \leq \mu^b = b^{-1}$ are not open at optimality no matter what is the total arrival rate to the system, as the outsourcing option is cheaper to use. If b is very large, in particular, if $b > 1/\mu_n$, and the arrival rate is large enough then all servers need to be open. Still, if the arrival rate is not too large then even if the outsourcing option is too expensive, we might see that that at optimality some slow servers are closed. For a given total arrival rate, as b increases the set of servers that are closed shrinks, and in fact for b large enough is it fixed at some set $\{1, \ldots, i^*(b)\}$. Let $z^* = (z_1^*, \ldots, z_n^*)$ be an optimal solution of Equation (10). We note that if $z_i^* \leq z^b(\mu_i)$ then z_i^* is the optimal rate of customers served by server $i \in N$, but if $z_i^* > z^b(\mu_i)$ then the rate of customers served by server i is $z^b(\mu_i)$ and the remaining rate of customers, namely $z_i^* - z^b(\mu_i)$, is outsourced. In general, an optimal solution of Equation (10) $z^* = (z_1^*, \ldots, z_n^*)$ means that the optimal routing to server $i \in N$ is $\min\{z_i^*, z^b(\mu_i)\}$. The following lemma proves that the optimal routing of customers among the servers is unique, and it also gives some structural properties that the vector z^* satisfies.

Lemma 2 *For the cooperative game $G = (N, V^b)$ defined in (10), there exists a unique optimal routing of customers to the servers. In addition, either for all $i \in N$ it holds that $z_i^* \geq z^b(\mu_i)$ or for all $i \in N$ it holds that $z_i^* < z^b(\mu_i)$ or $z_i^* = 0$.*

Proof: First note that the vector $(z_1^*, \ldots, z_n^*)$ that solves $V^b(N)$ is not necessarily unique. In spite of that we claim that the rate of customers that are routed to each server is unique. The proof is based on the form of the function ϕ. For a given server with capacity $\mu \geq \mu^b$, the cost function $\phi(\lambda, \mu)$ consists of two parts: one is the congestion cost that is paid for the customers that are served by the server (at optimality it is $\min\{z^b(\mu_i), z_i^*\}$), and the second is the cost paid for customers that are outsourced (at optimality it is $\max\{z_i^* - z^b(\mu_i), 0\}$). As the congestion cost part in $\phi(\lambda, \mu)$ that is denoted by $f(z, \mu) = \frac{z}{\mu - z}$ for $z \leq \lambda$ is strictly convex in $z < \mu$, the congestion that is assigned to each server at optimality is unique.

In addition, the convexity of ϕ implies that at optimality, the marginal cost of increasing congestion at a server is the same at all open servers, and this marginal cost is bounded from above by b. If this marginal cost is less than b then the outsourcing option is not exercised. Therefore, it is impossible that at optimality there exists two servers $i, j \in N$ with $z^b(\mu_j) > 0$, such that $z_i^* \geq z^b(\mu_i)$ and $z_j^* < z^b(\mu_j)$, as this means that the marginal cost of serving a customer at server j is lower than b, where z_i^* is already using the outsourcing option for its last customer. Such a solution can be improved by increasing z_j^* and decreasing z_i^* by the same small quantity. ■

In the rest of this subsection we show how the characteristic function $V^b(N)$ can be computed. We distinguish between two cases: first when the option where outsourcing is not utilized by coalition N under the optimal allocation, and second, when it is. In particular, note that if $\lambda(N) \geq \sum_{i=1}^{n} \mu_i$ then the outsourcing option must be used by the grand-coalition N no matter how large b is. The values $V^b(S)$, for any $S \subseteq N$, are computed in the same way.

Claim 2 *For a given b, the outsourcing option is not used by the grand-coalition N in the game $G = (V^b, N)$ if and only if $\lambda(N) \leq \sum_{i=1}^{n} z^b(\mu_i) = \sum_{\{i \in N \text{ and } \mu_i \geq 1/b\}} (\mu_i - \sqrt{\mu_i/b})$.*

Proof: The proof follows from the definition of $z^b(\mu_i)$ which is the maximum arrival rate that server i can serve at a marginal cost that is bounded from above by the outsourcing cost rate b. ∎

For given values b and $\lambda(N)$, regardless if the outsourcing option is exercised or not by the grand-coalition N, then it is not necessarily that all servers in $\{i \in N$ and $\mu_i \geq 1/b\}$ are open. In fact, some of the slow servers may be closed and their arrival stream is rerouted to faster server so as to minimize overall congestion. Indeed, given $\lambda(N)$ there exists a constant $\alpha(N)$ such that the outsourcing option is not used if and only if $b \geq \alpha(N)$. Later we specify the value of $\alpha(N)$.

- Case 1: The outsourcing option is not used by N. In this case it is optimal to serve the total arrival rate $\lambda(N)$ in-house. This model and its solution are described in [3] (see also [5], p.65). Accordingly, the last open server is

$$i^*(N) = \min\left\{ i \in N : \quad \mu_{i+1} \leq \frac{(\sum_{j=1}^{i} \mu_j - \lambda(N))^2}{(\sum_{j=1}^{i} \sqrt{\mu_j})^2} \right\},$$

the optimal congestion level is

$$V^b(N) = \frac{(\sum_{i=1}^{i^*(N)} \sqrt{\mu_i})^2}{\sum_{i=1}^{i^*(N)} \mu_i - \lambda(N)} - i^*(N)$$

and the optimal routing rate to any open server i, $1 \leq i \leq i^*(N)$ is

$$z_i^* = \mu_i - (\sum_{j=1}^{i^*(N)} \mu_j - \lambda(N)) \frac{\sqrt{\mu_i}}{\sum_{j=1}^{i^*(N)} \sqrt{\mu_j}}. \tag{12}$$

Note that as $0 \leq z_i < \mu_i$, $\frac{\partial f(z_i, \mu_i)}{\partial z_i} = \frac{\mu_i}{(\mu_i - z_i)^2}$. Since for $i \leq i^*(N)$, the corresponding $z_i^* > 0$, and hence the KKT conditions imply that

$\frac{\mu_i}{(\mu_i - z_i^*)^2} = \Theta \leq b$. Note the Θ is the Lagrance multiplier of the constraint $\Sigma_{i=1}^n z_i = \lambda(N)$. Also, as this derivative at zero equals $1/\mu_i$, we get that for $i^*(N) < i \leq N$, $1/\mu_i \geq \Theta$. Using Equation (12) for $i \leq i^*(N)$, results in

$$\Theta = \alpha(N) = \frac{(\sum_{k=1}^{i^*(N)} \sqrt{\mu_k})^2}{(\sum_{k=1}^{i^*(N)} \mu_k - \lambda(N))^2}. \tag{13}$$

Define $\alpha(S)$ for $S \subseteq N$ in a similar way.

The extreme case where no subcoalition of N uses the outsourcing option occurs if only if $\lambda(S) < \sum_{i \in S} \mu_i$ for any coalition $S \subseteq N$, and $b \geq \max_{S \subseteq N} \alpha(S) \equiv \bar{b}$. By using the ratio form of $\alpha(\cdot)$ in Equation (13), it is possible to show that $\bar{b} = \max\{\frac{\mu_i}{(\mu_i - \lambda_i)^2} : \ i \in N\}$. In this special case the vector z^* is unique as the function $f(z, \mu)$ is strictly convex in z, $0 \leq z < \mu$ and the core cost allocation based on competitive equilibrium prices allocates a cost $x_i = f(z_i^*, \mu_i) - \Theta(z_i^* - \lambda_i) = \frac{z_i^*}{\mu_i - z_i^*} - \Theta(z_i^* - \lambda_i)$ to server $i \in N$. That means that any server that is not open under the grand coaltion, i.e., $i \in \{i^*(N) + 1, \ldots, n\}$ pays $\Theta\lambda_i = \alpha(N)\lambda_i$ due to the service of his/her customers by other servers. An open server still pays for the congestion that he/she faces under the optimal allocation, but he/she is either compensated by Θ for an extra unit rate of arrivals (beyond λ_i) that he/she serves, or he/she needs to pay Θ per unit rate of arrivals of his/her λ_i that is routed to other servers. Equivalently, an open customer $i \leq i^*(N)$ pays $x_i = 2\sqrt{\mu_i}\sqrt{\alpha(N)} - \alpha(N)(\mu_i - \lambda_i) - 1$.

In the sequel (following the analysis of Case 2) we present a core cost allocation based on competitive equilibrium prices for the general case, i.e., when some coalitions use the outsourcing option and the others do not use it. The possibility of deriving a closed form core cost allocation for the general game is most pronounced since all we need to deal with in the grand-coalition. for which $\lambda(S) \geq \sum_{i \in S} \mu_i$. Identifying these coalition necessitates an exponential number of operations. Below we show how to overcome this issue.

- Case 2: The outsourcing option is used by N. In this case $\Theta = b$. Let $\mu_0 = \infty$ and

$$p(b) = \max\{k : k \geq 0, \ \mu_k^{-1} < b\}. \tag{14}$$

  - If $b \leq \mu_1^{-1}$, $p(b) = 0$, which means that no server is open and the total arrival rate of $\lambda(N)$ is outsourced. In fact, if $b \leq \mu_1^{-1}$ it is not profitable to serve any customer in-house (regardless of the coalition), thus $V^b(S) = b\lambda(S)$ for all $S \subseteq N$ and therefore there exists a single core cost allocation for this particular case, which is $x_i = b\lambda_i$ for $i \in N$.

– If $\mu_1^{-1} < b < \alpha(N)$, the set of open servers consists of the servers whose index is at most $p(b)$. Thus, for all $i \in N$, $\frac{\partial\phi(z_i,\mu_i)}{\partial z_i}|_{z_i^*} = b$. The solution in this case is not unique as the functions $\phi(z, \mu_i)$ is linear for $z > z^b(\mu_i)$. Any solution of the form $z_i^* = z^b(\mu_i) + \delta_i$, for $\delta_i \geq 0$, $i \in N$, that satisfies $\sum_{i\in N} z_i^* = \lambda(N)$ is optimal. Some algebra shows that the cost of the grand-coalition is

$$V^b(N) = b\lambda(N) - \sum_{i \leq p(b)} (1 - \sqrt{\mu_i b})^2 \tag{15}$$

As discussed above, also in this range not necessarily all subgames of (N, V^b) have their cost function of the form of equation (15), as some coalitions may not use the outsourcing option. In spite of that we point out a core cost allocation for any unobservable routing game with outsourcing.

The core allocation for the general case of $G = (N, V^b)$, which is based on competitive equilibrium prices uses Equation (11) that presents the game as a market game:

$$x_i = g_i(z_i^*) - \Theta(z_i^* - \lambda_i) \quad \text{for } i \in N \tag{16}$$

where $\Theta = \alpha(N)$ if the grand-coalition is not using the outsourcing option, and $\Theta = b$ if it does. As we see, Equation (16) provides a cost allocation that is in the core, where only the value for the grand-coalition, i.e., $V^b(N)$, needs to be solved.

For completeness we show that the game is not concave, ruling out the possibility of using Condition 1 in Section 1 for proving total balancedness.

Example 5 Consider $N = \{1, 2, 3\}$, with $\mu_1 = \mu_2 = 100$, $\lambda_1 = \lambda_2 = 1$, $\mu_3 = 1$ and $\lambda_3 = 0.99$. Assume that $b > \bar{b} = 10^4$ so outsourcing is not used by any coalition. Let $S = \{1, 3\}$ and $T = \{2, 3\}$. We have here $V^b(\{1\}) = V^b(\{2\}) = 0.01$, $V^b(\{3\}) = 99$. In coalition S server 1 is open and likewise server 2 is open in coalition T. Thus, $V^b(S) = V^b(T) = 0.02$. In coalition $S \cup T$ servers 1 and 2 are open, each getting half of the total traffic. Hence, $V^b(S \cup T) = 2(\frac{1+0.495}{100-1-0.495})$. It is easy to see $V^b(S \cap T) = V^b(\{3\}) = 99$. Hence, $V^b(S \cup T) + V^b(S \cap T) > V^b(S) + V^b(T)$, proving that the set function $V^b(\cdot)$ is not concave.

The next example shows a case where at least two servers are paid by the others under any core allocation, and hence no core allocation, which is either non-negative, or consisting of a single negative entry, exists. This is in contrast with our former paper [1], dealing with the model presented in Example 2, where we showed that under the type of cooperation defined there, core allocations with all its entries being non-negative always exist, and if there exist core cost allocations with negative entries, then there exist also core allocations with a single negative entry. Note that a negative entry means that a server, on top of his own customers being served by the grand coalition, is in fact being paid in order to join this coalition.

Example 6 Let $b > \bar{b}$, $n = 10$, $(\mu_1, \dots, \mu_{10}) = (100, 78, 70, 65, 50, 45, 30, 20, 10, 5)$ and $(\lambda_1, \lambda_2, \dots, \lambda_{10}) = (80, 60, 45, 20, 10, 20, 8, 12, 1, 4)$. $V(N) = 9.57$, which is attained when the first 8 servers are open. Let the vector $(x_1, \dots, x_{10})$ denote a core cost allocation. Let $\ell_i = V(N) - V(N \backslash \{i\})$ and note that for any game $x_i \geq \ell_i$, $i \in N$. In particular, $x_1 > \ell_1 = 3.09$. Addionally, by considering coalition $\{1, 4\}$, we get $x_1 + x_4 \leq 3.02$ and hence $x_4 \leq -0.07$. Likewise, by considering coalition $\{1, 5\}$, we get $x_1 + x_5 \leq 2.86$ and by the same reasoning we conclude that $x_5 \leq -0.23$. That means that any core allocation comes with both servers 4 and 5 being paid by the others.

5.2 A capacity sharing model

Consider a garage with a set of n service stations $N = \{1, \dots, n\}$. Each station provides maintenance to a particular brand of cars. Each station is modeled as an $M/M/1$ system, where station i is responsible for servicing a Poisson arrival rate of λ_i cars. Initially, station i is staffed by a team of workers so that its service time is exponentially distributed with parameter $\mu_i > \lambda_i$. The management is considering restaffing and reshuffling the existing manpower of capacity $\mu(N) = \sum_{i \in N} \mu_i$, among the stations in order to minimize cost. The management also considers better rationalization in the sense of the possible reduction of the capacity assigned to the systems, a step that may reduce manpower and salaries. The cost of a certain configuration of capacities is given by the total congestion cost minus the savings due to the capacity reduction. For simplicity assume that the unit cost of congestion is normalized to 1, and the savings per unit reduction in the capacity rate is $b \geq 0$. The respective game $G = (N, V^b)$ is formulated below as a regular market game, where each player $i \in N$ is associated with a single shared property, which is its surplus capacity $\mu_i - \lambda_i$, and a single individual property λ_i. Let $f(z_i, \lambda_i) = \frac{\lambda_i}{z_i}$ for $i \in N$, be the congestion cost at server $i \in N$ due to a surplus capacity of $z_i > 0$. Let

$$\phi(z, \lambda) = \min\{f(x, \lambda) - b(z - x) |\ 0 < x \leq z\} \tag{17}$$

denote the optimal cost of a server whose arrival rate is λ and its initial surplus capacity is z. It is easy to check that $\frac{\partial f(z,\lambda)}{\partial z} = -\frac{\lambda}{z^2}$, $z > 0$. Let $z^b(\lambda)$ be the value of z for which this derivative is equal to $-b$. Thus, $z^b(\lambda) = \sqrt{\frac{\lambda}{b}}$, and

$$\phi(z, \lambda) = \begin{cases} \frac{\lambda}{z} & \text{if } z \leq z^b(\lambda) \\ \frac{\lambda}{z^b(\lambda)} - b(z - z^b(\lambda)) & \text{otherwise.} \end{cases}$$

The cost of the grand-coalition is

$$V^b(N) = \min\{\sum_{i=1}^{n} \phi(z_i, \lambda_i) : \sum_{i=1}^{n} z_i = \mu(N) - \lambda(N) \text{ and } z_i \geq 0 \text{ for } 1 \leq i \leq n\}. \tag{18}$$

The values $V^b(S)$ for any $S \subset N$ are defined in the same way. The following Lemma is similar to Lemmas 1 and 2.

Lemma 3 *The capacity sharing game $G = (N, V^b)$ defined in (18) and (17) is a regular market game, and therefore it is totally balanced. In addition, there exists a unique optimal allocation of surplus capacities to the servers.*

Proof: We prove only the last part as the proof of the rest follows directly from the proof of Lemma 1 and the convexity of the functions $\phi(z_i, \lambda_i)$ in z_i for all $i \in N$. Similarly to the proof of Lemma 2 for the unobservable routing game with outsourcing, the vector $(z_1, \ldots, z_n)$ that solves (18) is not necessarily unique. However, the surplus capacities allocated to the servers, namely $(x_1, \ldots, x_n)$ (see Equation (17)) are unique in view of the fact that the function $f(z, \lambda)$ used in (17) is strictly convex. Thus, there exists a unique optimal reallocation of the total excess capacity among the $M/M/1$ systems with a possible reduction of the manpower.

■

In order to derive a core cost allocation for the capacity sharing game we distinguish between two cases: The first case, where the surplus capacity is not reduced, happens when b is small enough. Let $\beta(N)$ be a constant such that the surplus capacity for the grand-coalition N is not reduced if and only if $b \leq \beta(N)$. Later we will specify the value of $\beta(N)$.

- <u>Case 1: the surplus capacity is not reduced.</u> Under this case the total surplus capacity of $\mu(N) - \lambda(N)$ is distributed among the servers.

The optimal allocation of the surplus capacity and the optimal cost are described in [8]: $z_i^* = (\mu(N) - \lambda(N))\frac{\sqrt{\lambda_i}}{\sum_{j\in N}\sqrt{\lambda_j}}$, for $i \in N$, and

$$V^b(N) = \frac{(\sum_{i\in N}\sqrt{\lambda_i})^2}{\mu(N) - \lambda(N)}.$$

Note that under this case, for $i \in N$, $\frac{\partial\phi(z_i,\lambda_i)}{\partial z_i} = \frac{\partial f(z_i,\lambda_i)}{\partial z_i} = -\frac{\lambda_i}{z_i^2}$, $z_i > 0$. At optimality, there exists a constant Θ such that $-\frac{\lambda_i}{z_i^{*2}} = \Theta \leq -b$ as the option of reducing the surplus capacity is not used. Substituting into this last inequality the z_i^* values results in $\beta(N) = -\frac{(\sum_{i\in N}\sqrt{\lambda_i})^2}{(\mu(N)-\lambda(N))^2}$. There exists $\underline{b}$ such that for $b \leq \underline{b}$, the surplus capacity of any coalition $S \subseteq N$ is fully used internally. It is easy to show that $\underline{b} = \min_{S\subseteq N}\beta(S) = \min_{i\in N}\frac{\lambda_i}{(\mu_i-\lambda_i)^2}$. In this range the game (N, V^b) and all its sub-games do not use the capacity reduction option. The core cost allocation based on competitive equilibrium prices in this special case is $x_i = f(z_i^*, \lambda_i) - \beta(N)(z_i^* - (\mu_i - \lambda_i))$, $1 \leq i \leq n$. Some algebra leads to

$$x_i = 2\frac{\sqrt{\lambda_i}}{\sum_{j\in N}\sqrt{\lambda_j}}V^b(N) - \frac{\mu_i - \lambda_i}{\sum_{j\in N}(\mu_j - \lambda_j)}V^b(N),\ 1 \leq i \leq n.$$

This cost allocation is shown in [16] to be in the core of the game $G = (N, V^b)$ in a different way.

- Case 2: the surplus capacity is reduced. The marginal analysis for this case implies that $\Theta = -b$. This means that for any server $i \in N$ the optimal surplus capacity is $z^b(\lambda_i) = \sqrt{\frac{\lambda_i}{b}}$. The equalities $\frac{\partial\phi(z_i,\lambda_i)}{\partial z_i}|_{z_i^*} = -b$ for $i \in N$, imply that $z_i^* = z^b(\lambda_i) + \delta_i = \sqrt{\lambda_i/b} + \delta_i$, where $\delta_i \geq 0$ for $i \in N$, and the total reduction of capacity is $\sum_{i\in N}\delta_i = \mu(N) - \lambda(N) - \sum_{i\in N}\sqrt{\lambda_i/b}$. Substituting into (18) gives $\phi(z_i^*, \lambda_i) = \sqrt{\lambda_i b} - b\delta_i$, implying that

$$V^b(N) = 2\sqrt{b}\sum_{i\in N}\sqrt{\lambda_i} - b(\mu(N) - \lambda(N))\ .$$

The core allocation for the general case of $G = (N, V^b)$, which is based on competitive equilibrium prices uses Equation (18) that presents the game as a market game:

$$x_i = \phi(z_i^*, \lambda_i) - \Theta(z_i^* - (\mu_i - \lambda_i)) \ \text{ for } i \in N \tag{19}$$

where $\Theta = \beta(N)$ if the grand-coalition is not reducing its excess capacity, and $\Theta = -b$ in case it does. As we see, Equation (19) provides a cost allocation that is in the core, where only the optimization problem for the grand-coalition needs to be solved.

The next example shows that the game $G = (N, V^b)$ is non-concave.

Example 7 Using the same queueing system as stated in Example 5 with $b < \underline{b} = 1/99^2$, leading to never opting to save by reducing the surplus capacity under any coalition, results in a game which is not concave: The value of $V^b(\{1\})$ is large in comparison with the value of any other coalition so concavity is ruled out. Specifically, $V^b(S) = V^b(T) = 0.04$ where $V^b(S \cap T) = V^b(\{3\}) = 99$ and $V^b(S \cup T) > 0$. Hence, $V^b(S \cup T) + V^b(S \cap T) > V^b(S) + V^b(T)$, showing that V^b is a non-concave set function.

6 Conclusion

The main contribution of this paper is in addressing and formalizing the notion of regular games and their use. This class of games is quite large and includes many well-known games in operations management and queueing systems. We present two classes of such games: aggregation games and general resource sharing games. In particular, we focus on regular market games that are a subset of the second class: we provide sufficient conditions for a general resource sharing game to be a market game though it is not necessarily naturally presented as a market games. We propose a transformation that turns such a game into a market game, proving that it is totally balanced. Moreover, the transformation makes it possible to provide a core cost allocation for the game based on competitive equilibrium prices. Further investigation of regular games may yield new general interesting results on cooperative games.

In a forthcoming paper [2], we provide and prove a new sufficient condition which if satisfies by a regular game, then it is totally balanced. Specifically, we prove that a regular game that is sub-additive and *homogenous of degree 1* is totally balanced. Homogeneity of degree 1 means that given a set A of k vectors of properties, and a set $A^{(m)}$ that consists of m replicas of A, then $V(A^{(m)}) = mV(A)$. Apparently, regular market games turn out to be a proper sub-class of sub-additive and homogenous of degree 1 games, but the class of sub-additive and homogenous of degree 1 games contains many additional games. This in no way makes the identification of regular market games redundant, as for such games we point out a specific core cost allocation, where this cannot be done for a general sub-additive and homogenous of degree 1 game.

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