MATCHING OF LIKE RANK AND THE SIZE OF THE CORE IN THE MARRIAGE PROBLEM

by

R. Holzman†
D. Samet ‡

Working Paper No 25/2012 December 2012

Research No. 06590100

† Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel. E-mail: holzman@tx.technion.ac.il

‡ Faculty of Management, Tel Aviv University, Tel Aviv 69978, Israel.
E-mail: dovsamet@gmail.com

Helpful discussions with Yael Deutsch, Vicki Knoblauch, Alvin Roth, and the late Uriel Rothblum are gratefully acknowledged.

This paper was partially financed by the Henry Crown Institute of Business Research in Israel.

Matching of like rank and the size of the core in the marriage problem *

Ron Holzman† and Dov Samet‡

Abstract

When men and women are objectively ranked in a marriage problem, say by beauty, then pairing individuals of equal rank is the only stable matching. We generalize this observation by providing bounds on the size of the rank gap between mates in a stable matching in terms of the size of the ranking sets. Using a metric on the set of matchings, we provide bounds on the diameter of the core—the set of stable matchings—in terms of the size of the ranking sets and in terms of the size of the rank gap. We conclude that when the set of rankings is small, so are the core and the rank gap in stable matchings.

1 Introduction

1.1 Matching of likes

When considering the dazzling world of stardom and glamor we are not at all surprised to see Angelina Jolie and Brad Pitt as a couple. Both are highly ranked in this world, and their match seems natural. We would be bewildered, on the other hand, to see Jolie matched up with another man of this world whose physical appearance ranks much lower than hers. Such a man, so we expect, would be naturally matched with a woman ranked like him.

Those who are not familiar with the world of entertainment, may find it easier to relate to a similar mating of likes in the academic arena. Highly ranked scholars are affiliated, more often than not, with top-tier universities, while those who are academically less attractive are affiliated with lesser universities.

1.2 Matching of like trait and of like rank

The phenomenon of matched people being similar in terms of traits like beauty, intellect, and education is called (positive) assortative matching. The model

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†Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel. E-mail: holzman@tx.technion.ac.il

‡Faculty of Management, Tel Aviv University, Tel Aviv 69978, Israel. E-mail: dovsamet@gmail.com

used by Becker (1973) to study the economic theory of marriage gives rise, under certain conditions, to assortative matching. The main elements in this model are the quantified traits of individuals and the production function that associates with the traits of a pair of individuals of different gender the output produced when they are matched.

Here we examine the phenomenon of matching likes in the elementary marriage problem introduced by Gale and Shapley (1962). Unlike Becker's model, in a marriage problem, individuals are not endowed with objective traits, and their preferences are not expressed in terms of an objective cardinal production function. Rather, *subjective* preferences are given by specifying for each individual an ordinal ranking of the opposite gender. The solution for a marriage problem is its core, which consists of the stable matchings. A matching is stable if there is no pair of individuals of different gender who are not matched to each other, but prefer to be matched to each other than to their mates. Since individuals in a marriage problem do not have traits, we cannot describe the phenomenon of matching of likes as matching of like traits. Instead we describe it as matching of like rank, which requires some clarification.

1.3 The case of a universal ranking

Although the ranking of individuals in a marriage problem is not necessarily derived from some objective trait, there is a simple case in which a marriage problem can be thus interpreted, namely, the case that rankings are *universal.* That is, all men rank women in the same way and all women rank men in the same way. Here we can say that the ranking reflects an objective measuring of a trait.

The analysis of this case is simple. Matching pairs of equal rank is the only stable matching for such a problem. Moreover, even if only one side, say the men, are universally ranked there still exists only one stable matching. In this case each man is matched to a woman whom *he* ranks at least as high as his objective rank.

We can conclude that universal ranking is associated with the smallest possible core, and the equality of rank of matched individuals.

1.4 Correlated rankings

Obviously, the assumption of a universal ranking is too strong. We can hardly expect a unanimous, universal ranking in anything that involves human beings. Our purpose here is to quantify the notion of "matching of like rank" so that we can say for *any* marriage problem to what extent individuals are matched with their likes in stable matchings.

We can expect matching of likes to the extent that we can identify some objective component in the rankings, that is, if individual rankings are positively correlated which reflects a partial agreement on some hidden trait. This may indeed often be the case. Beauty, for example, is indeed in the eyes of the beholder. Nevertheless, in a given culture there is a great deal of agreement in the judgement of beauty, and rankings of beauty are positively correlated. Similarly, scholars may differ on the ranking of universities, but in all rankings Harvard is among the top, say, ten universities. Here, we use the bound on the range of ranks of each individual in a set of rankings as a measure of the positive correlation of the rankings in this set. The smaller the bound the more objective are the rankings. Thus, we expect that the smaller this bound, the greater is the similarity between matched pairs in stable matchings, and the smaller is the core.

1.5 Like rank and the size of the core

The notion of like rank is obvious in the case of universal ranking on both genders, because rank in this case is uniquely defined. But when ranking is not universal there is no objective rank to serve as a basis for comparison. Instead we adopt a subjective measure of like rank. The alikeness of rank of two matched individuals is measured by the difference between their ranks *according to the rankings of these two individuals*. We call the absolute value of this difference the *rank gap* for this pair of individuals.[1] To the extent that individual rankings are close and therefore reflect an objective component, the rank gap serves as proxy to the comparison of objective ranks.

The size of the core of a marriage problem can be easily measured. From the point of view of a woman, the core is small if the ranks of the men she is matched to in the two optimal stable matchings are close. The core is small for the women if it is small for them on average. A similar measure of the size of the core can be defined for men.

1.6 The main results

Equipped with precise definitions of the above mentioned measures we give here bounds on the size of the core and on the rank gap in terms of the size of the sets of rankings. In particular it follows that when these sets are small, that is, when rankings are close to being objective, then the set of stable matchings is small and the rank gap is also small. We show, moreover, that the size of the core has a bound in terms of the rank gap of the two optimal stable matchings.

1.7 Related work

The empirical relation between positively correlated rankings and the size of the core in the college admission problem, was observed by Roth and Peranson (1999):

> "One factor that strongly influences the size of the set of stable matchings (which coincides with the core in this simple model) is the

[1] If we think of the ranking as ordering the opposite gender according to "love" then the rank gap of a matched pair measures precisely the term "*more*" when one of them complains to the other "I love you *more* than you love me".

> correlation of preferences among programs and among applicants. When preferences are highly correlated (i.e., when similar programs tend to agree which are the most desirable applicants, and applicants tend to agree which are the most desirable programs), the set of stable matchings is small."

The theoretical result here is one possible formalization of this observation.

Caldarelli and Capocci (2001) and Boudreau and Knoblauch (2010) studied correlation of rankings via statistical simulation. They introduce an objective trait of agents measured numerically, and assigning the value of this trait to individuals by random variables, they generate correlated and intercorrelated rankings. The simulations are restricted to the optimal matchings obtained by the deferred acceptance algorithm. Their main interest is in gender satisfaction, which is the sum of the ranks of the women by their mates in the optimal matchings.

Eeckhout (2000) and Clark (2006) gave conditions on the preferences that are sufficient for the uniqueness of stable matchings. However, they did not investigate conditions under which the set of stable matchings, though not necessarily a singleton, must be small.

2 Preliminaries

A ***ranking*** of a nonempty finite set X is a bijection $r : X \to \{1, \ldots, |X|\}$. If $r(x) < r(x')$ we say that r ranks x *higher* than x'.

A ***marriage market*** is a tuple (M, W, R_M, R_W) where M and W are disjoint sets of finite size $n > 0$ of ***men*** and ***women***, called the two ***sides*** of the market, $R_M = (r_m)_{m \in M}$ is an n-tuple of rankings of W by the men, and $R_W = (r_w)_{w \in W}$ is an n-tuple of rankings of M by the women. We refer also to R_M and R_W as the *sets* of rankings in each n-tuple correspondingly. No confusion will result.

A ***matching*** is a set of pairs $\mu = \{(m, w)\}$ which is the graph of a bijection of M and W. For each man m we denote by $\mu(m)$ the unique woman w such that $(m, w) \in \mu$. For each woman w, $\mu(w)$ is similarly defined.

A pair (m, w) ***blocks*** the matching μ if $r_m(w) < r_m(\mu(m))$ and $r_w(m) < r_w(\mu(w))$. The matching μ is ***stable*** if no pair blocks it. The core, C, of the marriage problem is the set of all its stable matchings. There exists a man-optimal stable matching, μ_M, that satisfies for each m and w, $r_m(\mu_M(m)) = \min_{\mu \in C} r_m(\mu(m))$, and $r_w(\mu_M(w)) = \max_{\mu \in C} r_w(\mu(w))$. Similarly, there exists a woman-optimal stable matching μ_W with the corresponding properties.

3 From universal ranking to general ranking

We say that one of the sides of a marriage problem is ***universally ranked*** if it is ranked in the same way by all the individuals of the other side. If, say, the men are universally ranked as $m_1, \ldots, m_n$, then it is easy to check that in any stable matching m_1 must be matched to his top choice, m_2 must be matched to his highest choice among the remaining $n-1$ women, and so on. This leads to the following consequences:

(i) If one of the sides, say M, is universally ranked, then there exists a unique stable matching.

(ii) In this matching each man ranks his spouse at least as high as she ranks him.

(iii) Consequently, when both sides are universally ranked, then individuals who are matched in the unique stable matching, have the same rank.

In this section we generalize these three results by relating each pair in the following list:

- The size of the sets of rankings;
- the size of the gap between the ranks that spouses rank each other in stable matchings;
- the size of the core.

In particular, it follows that when rankings are close the set of stable matchings is small and the gap between the ranks of the individuals who are matched in a stable matching is also small.

3.1 The ranking sets and the rank gap

Given a set R of rankings of a set X, the ***displacement*** of $x \in X$ is $\delta(x) = \max_{r\in R} r(x) - \min_{r\in R} r(x)$. We use the maximal displacement, $\Delta^{\max}(R) = \max_{x\in X} \delta(x)$ and the average displacement, $\Delta^{\text{av}}(R) = (1/n)\sum_{x\in X} \delta(x)$ as measures of the size of R.

The ***rank gap*** of a pair $(m,w) \in M\times W$ is $\gamma(m,w) = |r_m(w) - r_w(m)|$. The disparity of the mutual rankings of spouses in a given matching μ is measured by the maximal rank gap in μ, $\Gamma^{\max}(\mu) = \max_{(m,w)\in\mu} \gamma(m,w)$ and the average rank gap in μ, $\Gamma^{\text{av}}(\mu) = (1/n)\sum_{(m,w)\in\mu} \gamma(m,w)$.

The next theorem generalizes (ii).

Theorem 1 *For each stable matching μ and $(m,w) \in \mu$,*

$$r_m(w) - r_w(m) \leq 2\Delta^{\max}(R_W). \tag{1}$$

Proof: Let μ be a stable matching and $(m,w) \in \mu$. Man m ranks $r_m(w)-1$ women higher than w. By the stability of μ each one of these $r_m(w)-1$ women is matched to a man she ranks higher than m. We now compute an upper bound on the number of men that can be ranked higher than m by at least one woman. Obviously, $r_m(w)-1$ cannot exceed such an upper bound.

Let $r_{w_0}(m) = \max_{w' \in W} r_{w'}(m)$. If a man is ranked higher than m by some woman, then his rank is $r_{w_0}(m) - 1$ or higher. For each man m' and woman w', $|r_{w'}(m') - r_{w_0}(m')| \leq \delta(m')$. Thus if $r_{w_0}(m') > r_{w_0}(m) - 1 + \delta(m')$, then for every woman w', $r_{w'}(m') > r_{w_0}(m) - 1$. Thus, all men ranked by w_0 lower than $r_{w_0}(m) - 1 + \Delta^{\max}(R_W)$ cannot be ranked by any woman $r_{w_0}(m) - 1$ or higher. Thus, at most $r_{w_0}(m) - 1 + \Delta^{\max}(R_W)$ men can be ranked above m.[2] As at least $r_m(w) - 1$ men are ranked above m we conclude

$$r_m(w) - 1 \leq r_{w_0}(m) - 1 + \Delta^{\max}(R_W). \tag{2}$$

Also,

$$r_{w_0}(m) - r_w(m) \leq \delta(m). \tag{3}$$

Adding (2) and (3) we have $r_m(w) - r_w(m) \leq \Delta^{\max}(R_W) + \delta(m) \leq 2\Delta^{\max}(R_W)$. ∎

Claim (ii) is a special case of this theorem for $\Delta^{\max}(R_W) = 0$. We next generalize claim (iii).

Corollary 1 *For any stable matching μ,*

$$\Gamma^{\max}(\mu) \leq 2\max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)\}. \tag{4}$$

Proof: By (1) and the analogous bound $r_w(m) - r_m(w) \leq 2\Delta^{\max}(R_M)$ we conclude that for each stable matching μ and pair $(m, w) \in \mu$, $|r_w(m) - r_m(w)| \leq 2\max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)\}$, from which (4) follows. ∎

By this corollary, when ranking is universal on both sides, the rank gap of any stable matching vanishes, as claimed in (iii). This also implies the uniqueness of the stable matching, since when rankings are universal there is only one matching for which the rank gap vanishes.

When the maximal displacement is much larger than the average one, the upper bound on the maximal rank gap obtained above may not be useful. Hence we proceed to establish an upper bound in terms of average displacements.

Theorem 2 *For every stable matching μ and any subset M_0 of M,*

$$\sum_{m \in M_0} [r_m(\mu(m)) - r_{\mu(m)}(m)] \leq \sum_{m \in M_0} \delta(m) + \sum_{m' \in M} \delta(m') \leq 2n\Delta^{\text{av}}(R_W). \tag{5}$$

Proof: Let μ be a stable matching and $M_0 \subseteq M$. Consider a man $m \in M_0$. As in the proof of Theorem 1, by the stability of μ, we can find $r_m(\mu(m)) - 1$ men m', each of them satisfying

$$r_{\mu(m')}(m') < r_{\mu(m')}(m). \tag{6}$$

[2] For $\Delta^{\max}(R_W) = k \geq 1$ a tighter bound holds. At most $r_{w_0}(m) - 2 + k$ men can be ranked above m, because m is among the $r_{w_0}(m) - 1 + k$ men ranked highest by w_0. Thus, for $k \geq 1$ the bound of (1) can be improved to $2\Delta^{\max}(R_W) - 1$. For $k = 0$, m is not included among the $r_{w_0}(m) - 1 + k$ highest ranked men.

Given any fixed woman and any rank i, she can rank at most $i-1$ of those men m' in ranks $1, \ldots, i-1$. Taking $i = \max_{w \in W} r_w(m)$, we conclude that at least $r_m(\mu(m)) - \max_{w \in W} r_w(m)$ of the men m' satisfy

$$\max_{w \in W} r_w(m') \geq \max_{w \in W} r_w(m). \tag{7}$$

Denote by P_m the set of men m' satisfying (6) and (7). As shown, we have $|P_m| \geq r_m(\mu(m)) - \max_{w \in W} r_w(m)$. Doing this for each $m \in M_0$ separately, we get a system of sets P_m, $m \in M_0$, with union $P = \cup_{m \in M_0} P_m$. For each $m' \in P$, let $Q_{m'} = \{m \in M_0 | \, m' \in P_m\}$. Such a man m' satisfies (6) with respect to every $m \in Q_{m'}$, and therefore

$$r_{\mu(m')}(m') \leq \max_{m \in Q_{m'}} r_{\mu(m')}(m) - |Q_{m'}| \leq \max_{m \in Q_{m'}} \max_{w \in W} r_w(m) - |Q_{m'}|. \tag{8}$$

On the other hand, since m' satisfies (7) with respect to every $m \in Q_{m'}$, we get

$$\max_{w \in W} r_w(m') \geq \max_{m \in Q_{m'}} \max_{w \in W} r_w(m). \tag{9}$$

Combining (8) and (9), we obtain that $\delta(m') \geq |Q_{m'}|$. This yields

$$\begin{aligned} \sum_{m \in M_0} [r_m(\mu(m)) - \max_{w \in W} r_w(m)] &\leq \sum_{m \in M_0} |P_m| = \sum_{m' \in P} |Q_{m'}| \\ &\leq \sum_{m' \in P} \delta(m') \leq \sum_{m' \in M} \delta(m'). \end{aligned} \tag{10}$$

We also have

$$\sum_{m \in M_0} [\max_{w \in W} r_w(m) - r_{\mu(m)}(m)] \leq \sum_{m \in M_0} \delta(m), \tag{11}$$

and upon adding (10) and (11) we get (5). ∎

Corollary 2 *For any stable matching μ,*

$$\Gamma^{\text{av}}(\mu) \leq 2(\Delta^{\text{av}}(R_W) + \Delta^{\text{av}}(R_M)). \tag{12}$$

Proof: Given a stable matching μ, let M_0 be the set of men m for whom $r_m(\mu(m)) > r_{\mu(m)}(m)$, and let W_0 be the set of women w for whom $r_w(\mu(w)) > r_{\mu(w)}(w)$. Adding (5) and the analogous bound for the subset W_0 of W, and dividing by n, we obtain (12). ∎

3.2 The size of the core

We now provide bounds on the size of the core in terms of the size of the ranking sets and the rank gap. For this we define two metrics on matchings. The ***woman-metric*** on matchings, d_W, is defined for each pair of matchings μ_1 and μ_2 by

$$d_W(\mu_1, \mu_2) = (1/n) \sum_{w \in W} |r_w(\mu_1(w)) - r_w(\mu_2(w))|.$$

The ***man-metric*** d_M is similarly defined. The diameters of the core with respect to the metrics d_W and d_M are denoted by $D_W(C)$ and $D_M(C)$ correspondingly. For stable matchings μ_1 and μ_2, $|r_w(\mu_1(w)) - r_w(\mu_2(w))| \leq r_w(\mu_M(w)) - r_w(\mu_W(w))$ for each w. Thus, $D_W(C) = (1/n)\sum_{w\in W}[r_w(\mu_M(w)) - r_w(\mu_W(w))]$, and a similar expression holds for $D_M(C)$.

The following theorem generalizes (i).

Theorem 3

$$D_W(C) \leq \Delta^{\rm av}(R_W).$$

Proof:

$$\begin{aligned}
D_W(C) &= (1/n)\sum_{w\in W}[r_w(\mu_M(w)) - r_w(\mu_W(w))] \\
&= (1/n)\sum_{m\in M}[r_{\mu_M(m)}(m) - r_{\mu_W(m)}(m)] \\
&\leq (1/n)\sum_{m\in M}\delta(m) \\
&= \Delta^{\rm av}(R_W).
\end{aligned}$$

∎

When one side of the market is universally ranked, then by Theorem 3, $D_W(C) = 0$ or $D_M(C) = 0$, and in either case C is a singleton. Thus, claim (i) is a special case of the theorem.

In the next theorem, the size of the core is bounded in terms of the average gap of the woman and man optimal matchings.

Theorem 4

$$D_M(C) + D_W(C) \leq \Gamma^{\rm av}(\mu_M) + \Gamma^{\rm av}(\mu_W).$$

Proof: Define $S_{MM} = \sum_{m\in M} r_m(\mu_M(m))$ and $S_{MW} = \sum_{m\in M} r_m(\mu_W(m))$, and define S_{WW} and S_{WM} similarly. Then $D_M(C) = (1/n)[S_{MW} - S_{MM}]$ and $D_W(C) = (1/n)[S_{WM} - S_{WW}]$. Next, observe that

$$\begin{aligned}
|S_{WM} - S_{MM}| &= |\sum_{w\in W} r_w(\mu_M(w)) - \sum_{m\in M} r_m(\mu_M(m))| \\
&= |\sum_{w\in W} r_w(\mu_M(w)) - \sum_{w\in W} r_{\mu_M(w)}(w)| \\
&\leq \sum_{w\in W} |r_w(\mu_M(w)) - r_{\mu_M(w)}(w)| \\
&= n\Gamma^{\rm av}(\mu_M),
\end{aligned}$$

and similarly, $|S_{MW} - S_{WW}| \leq n\Gamma^{\rm av}(\mu_W)$. Thus,

$$\begin{aligned} D_M(C) + D_W(C) &= (1/n)[S_{MW} - S_{MM} + S_{WM} - S_{WW}] \\ &\le (1/n)[|S_{WM} - S_{MM}| + |S_{MW} - S_{WW}|] \\ &\le \Gamma^{\text{av}}(\mu_M) + \Gamma^{\text{av}}(\mu_W). \end{aligned}$$

■

The following is an immediate corollary of this theorem.

Corollary 3 *If the rank gaps in the man-optimal and the woman-optimal matchings vanish, then there exists a unique stable matching.*

3.3 Examples and counterexamples

We present here a few constructions of marriage markets, showing that some of the bounds proved above are tight, and indicating that certain variants of these bounds do not hold in general.

Our first example shows that the upper bounds in Theorem 1 and Corollary 1 are tight (in the slightly improved form given in footnote 2).

Example 1 Let $k \ge 1$. Consider a market with $2k$ individuals on each side, numbered as $M = \{m_1, \dots, m_{2k}\}$ and $W = \{w_1, \dots, w_{2k}\}$. Let the women be universally ranked from top to bottom as $w_1, \dots, w_{2k}$. Let the rankings of the men by the women be as follows:

$$\begin{aligned} w_i &: m_1, m_2, \dots, m_k, m_{2k}, m_{k+1}, \dots, m_{2k-1} \quad (i = 1, \dots, k) \\ w_j &: m_1, m_{k+1}, \dots, m_{2k-1}, m_{2k}, m_2, \dots, m_k \quad (j = k+1, \dots, 2k-1) \\ w_{2k} &: m_{2k}, m_1, \dots, m_{k-1}, m_k, m_{k+1}, \dots, m_{2k-1} \end{aligned}$$

The unique stable matching is obtained when each of the women $w_1, \dots, w_{2k}$ in turn gets her top choice among the still available men. This yields the matching $\{(m_i, w_i)\}_{i=1,\dots,2k}$, with $r_{m_{2k}}(w_{2k}) - r_{w_{2k}}(m_{2k}) = 2k - 1$. On the other hand, it is easy to check that $\Delta^{\max}(R_W) = k$. This shows that the upper bound $r_m(w) - r_w(m) \le 2\Delta^{\max}(R_W) - 1$ (for $\Delta^{\max}(R_W) \ge 1$) is tight. As $\Delta^{\max}(R_M) = 0$, this example also shows that one cannot replace the upper bound $2\max\{\Delta^{\max}(R_W), \Delta^{\max}(R_M)\}$ on $\Gamma^{\max}(\mu)$ by $\Delta^{\max}(R_W) + \Delta^{\max}(R_M)$.

For the bound on the average rank gap in terms of the average displacements, we do not have a construction meeting the upper bound. In fact, we conjecture that the factor of 2 in the upper bounds of Theorem 2 and Corollary 2 can be lowered to 1. The following example shows that it cannot be replaced by any constant factor smaller than 1.

Example 2 Consider a market with $M = \{m_1, \dots, m_n\}$ and $W = \{w_1, \dots, w_n\}$. Let the women be universally ranked as $w_1, \dots, w_n$. Let the ranking of the men by woman w_i, $i = 1, \dots, n$, be obtained from the ranking $m_1, \dots, m_n$

by promoting m_i to the top of the list, while leaving the other men in the same order. The unique stable matching is $\{(m_i, w_i)\}_{i=1,\ldots,n}$. Here the rank gaps are $0, 1, \ldots, n-1$ respectively, while the displacements of the men are $1, 2, \ldots, n-1, n-1$ respectively. Thus $\Gamma^{\text{av}}(\mu) = (n-1)/2$, $\Delta^{\text{av}}(R_W) = (n+2)(n-1)/(2n)$, and the ratio between them approaches 1 as n goes to infinity.

According to statement (i) above, if *either one* of the sides is universally ranked, then the core is a singleton. Thus, one may expect to be able to assert that the diameter of the core in the woman-metric, $D_W(C)$, is small, not only when $\Delta^{\text{av}}(R_W)$ is small (as shown in Theorem 3), but also when $\Delta^{\text{av}}(R_M)$ is small. The following example refutes this intuition, and illustrates some additional points that we discuss below.

Example 3 Let $k \geq 2$, and let n be a multiple of k, say $n = k\ell$. Consider a market where the men are partitioned into ℓ blocks of size k each: $M^i = \{m_1^i, \ldots, m_k^i\}$, $i = 1, \ldots, \ell$. Similarly, the women are partitioned into $W^i = \{w_1^i, \ldots, w_k^i\}$, $i = 1, \ldots, \ell$. Let every man rank the blocks of women as $W^1, \ldots, W^\ell$; within the blocks, the women are ranked as $w_1^i, \ldots, w_k^i$, except that for each i, the men in M^i rank the women in the corresponding block W^i in a cyclic fashion:

$$m_j^i : w_j^i, \ldots, w_k^i, w_1^i, \ldots, w_{j-1}^i$$

(with subscripts taken modulo k). Every woman in W^i, $i = 1, \ldots, \ell$, ranks $\cup_{p=1}^i M^p$ above the rest of the men; within this union of blocks, woman w_j^i ranks m_{j+1}^i first and m_j^i last (that is, in rank ik); besides that, the rankings are immaterial.

One may check, by induction on i, that in every stable matching the men in M^i are matched to the women in W^i, $i = 1, \ldots, \ell$. Within each pair of blocks M^i, W^i, the man-optimal stable matching μ_M consists of the pairs $\{(m_j^i, w_j^i)\}_{j=1,\ldots,k}$, whereas the woman-optimal one μ_W consists of the pairs $\{(m_{j+1}^i, w_j^i)\}_{j=1,\ldots,k}$. Thus,

$$D_W(C) = \frac{1}{k\ell} \sum_{i=1}^{\ell} k(ik-1) = \frac{k(\ell+1)}{2} - 1.$$

Note that $\Delta^{\text{av}}(R_M) = \Delta^{\max}(R_M) = k-1$. By keeping k fixed and letting ℓ grow, we see that $D_W(C)$ cannot be bounded by any function of $\Delta^{\text{av}}(R_M)$ or even $\Delta^{\max}(R_M)$. As remarked above, if $\Delta^{\max}(R_M)$ vanishes then so does $D_W(C)$, but our construction shows that any positive value of $\Delta^{\max}(R_M)$ is consistent with arbitrarily large values of $D_W(C)$.

We observe also that in our example $D_M(C) = k-1$, which shows (upon interchanging the roles of men and women) that Theorem 3 is tight. It may also be checked that our example gives equality in Theorem 4, thus showing its tightness, as well.

Example 3 serves to illustrate yet another point. A different way to measure the size of a set R of rankings of a set X would be to define a metric on R

by $d(r,r') = (1/n)\sum_{x\in X}|r(x)-r'(x)|$, and consider $D(R) = \max_{r,r'\in R} d(r,r')$, the diameter of R under this metric. Note that in general $D(R) \le \Delta^{\rm av}(R)$. To calculate $D(R_M)$ in our example, note that if $m \in M^i$ and $m' \in M^{i'}$ then r_m and $r_{m'}$ may differ only regarding women in $W^i \cup W^{i'}$. Within W^i one ranking is a cyclic shift of the other, so in the worst case we have $\sum_{w\in W^i}|r_m(w)-r_{m'}(w)| = \lfloor k^2/2\rfloor$, and similarly for $W^{i'}$. This gives

$$D(R_M) = \frac{1}{k\ell}\cdot 2\lfloor\frac{k^2}{2}\rfloor \le \frac{k}{\ell}.$$

By keeping the ratio k/ℓ fixed while both of them grow, we see that $D(R_M)$ can be arbitrarily small while $D_M(C)$ is arbitrarily large. We conclude that this alternative measure of the size of a set of rankings cannot replace $\Delta^{\rm av}$ in providing an upper bound on the size of the core (or, for that matter, on the average rank gap).

Our final question is whether there exist upper bounds, similar to Theorems 3 and 4, not only on the rank difference between mates in μ_M and μ_W for an average individual, but for *every* individual. The following example gives a negative answer.

Example 4 Consider a market with $M = \{m_1,\ldots,m_n\}$ and $W = \{w_1,\ldots,w_n\}$. The women are basically ranked as $w_2,\ldots,w_n,w_1$, but man m_i, $i = 2,\ldots,n-1$, swaps w_i and w_{i+1} in his ranking, and the other two men make specific adjustments as indicated:

$$\begin{array}{rcll} m_1 & : & w_2, w_1, w_3, \ldots, w_n & \\ m_i & : & w_2,\ldots,w_{i-1},w_{i+1},w_i,w_{i+2},\ldots,w_n,w_1 & (i=2,\ldots,n-1) \\ m_n & : & w_2,\ldots,w_{n-1},w_1,w_n & \end{array}$$

The men are basically ranked as $m_1,\ldots,m_n$, but woman w_i, $i = 2,\ldots,n$, swaps m_{i-1} and m_i in her ranking, yielding the rankings:

$$\begin{array}{rcll} w_1 & : & m_1, m_2, \ldots, m_n & \\ w_i & : & m_1,\ldots,m_{i-2},m_i,m_{i-1},m_{i+1},\ldots,m_n & (i=2,\ldots,n) \end{array}$$

We claim that $\mu = \{(m_i, w_{i+1})\}_{i=1,\ldots,n-1} \cup \{(m_n, w_1)\}$ is the man-optimal stable matching. To check stability, note that if m_i prefers w_j to his mate then $2 \le j \le i-1$, but such a woman w_j prefers her mate to m_i. To verify that μ is man-optimal, use the fact that μ_M must satisfy $r_w(\mu_M(w)) \ge r_w(\mu(w))$ for every $w \in W$. Considering in turn the women $w_1, w_n, w_{n-1}, \ldots$, this forces $\mu_M = \mu$.

Next, we claim that $\mu' = \{(m_i, w_i)\}_{i=1,\ldots,n}$ is the woman-optimal stable matching. To check stability, note that if w_i prefers m_j to her mate then $j \le i-2$, but such a man m_j prefers his mate to w_i. To verify that μ' is woman-optimal, use the property $r_w(\mu_W(w)) \le r_w(\mu'(w))$ successively for the women $w_1, w_2, w_3, \ldots$, deducing that $\mu_W = \mu'$.

Now, woman w_1 is matched to her top-ranked man m_1 in μ_W and to her bottom-ranked man m_n in μ_M. This is in spite of the fact that $\delta(m) \leq 2$ for every $m \in M$, which shows that the upper bound of Theorem 3 does not hold when both sides of the inequality are replaced by their max versions. A similar conclusion applies to the bound of Theorem 4, since $\Gamma^{\max}(\mu_M) = 2$ and $\Gamma^{\max}(\mu_W) = 1$.

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