OPTIMAL PRICING AND REPLENISHMENT POLICY IN A MARKET WITH TWO TYPES OF CUSTOMERS WHO HOLD INVENTORY

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Abstract

We consider a pricing and replenishment model in which the retailer advertises the selling periods of the product at the beginning of the time horizon. The customers differ in their reservation price and the time they most need the product. The customers are assumed to be discretionary, meaning that if at their most desired time the product is out of stock then they may decide to buy it earlier or later than that time, paying a holding or shortage cost, respectively, or they may decide to quit. Such a modeling of the customers' behavior induces a partial backlogging system in which the retailer is exempted from the burden of having to estimate his/her backlogging or lost-sales cost parameters. We investigate the retailer's optimal policy for two types of customers, each associated with its own arrival rate and reservation price.

Keywords: Inventory/production: pricing, review, deterministic.

1 Introduction

Inventory models customarily assume that clients who do not find the product on the shelf upon their arrival, either quit (lost sales) or wait for the next reorder time (backlogging). In this paper we follow Glazer and Hassin (1986,1990) and consider a variation of the classical models, in which a customer can also buy the product earlier than the time she most desires it. Such a model is possible when the retailer advertises beforehand the exact selling periods. A customer who buys earlier than needed is subject to inventory holding costs; a customer who buys later than needed is subject to shortage costs. A customer's decision of whether and when to buy the

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product is solely based on the criterion of maximizing her net gain. The retailer, who knows the utility function of the customers, may avoid selling the product in certain periods if this raises his net average profit. By doing so, the retailer manipulates some of the customers to advance or postpone their purchase.

As an example for such a model, consider a car dealer who sells a certain exclusive model non-continuously. Customers who know that the product will not be available at the time they most desire it, may decide to buy it earlier than needed (incurring financial charges), or alternatively they may decide to wait and buy the car as soon as it is back in stock, paying the rental cost of an equivalent car.

Glazer and Hassin (1986) solved this model assuming that customers are identical in all but the time they need the good. Their main qualitative finding is that the solution may be of one of three types: Continuous sales throughout the cycle, sales only at the time of inventory replenishment, or continuous sales through an interval followed by a *no-stock interval*, i.e. an interval in which the firm does not hold stock, that ends with the next replenishment. This finding adds to classic models by explaining real cases in which sellers do not hold inventory at all.

The assumption of identical customers greatly limits the applicability of the above mentioned results. In this paper we extend the model and identify the optimal integrated pricing, replenishment, and selling schedule policy for two types of customers, each associated with its own constant arrival rate and reservation price. More specifically, we consider an infinite-horizon, continuous-time, deterministic, and stationary pricing and inventory model of a single-type product sold by a monopolist retailer who advertises at the beginning of the horizon its fixed price and the sales-intervals during which it is sold. The retailer faces a setup cost per order and a linear inventory holding cost. Each customer is characterized by a reservation price and the time she most needs the product. A customer's net gain from buying the good is the difference between her reservation price and her expenses due to the cost of the product and her holding/shortage costs if she buys it at a time which is not her most desired time. Thus, only customers whose net gain is non-negative buy the product. Assuming, in addition, a price-sensitive demand, the retailer is faced by a backlogging/shortage mechanism, in which in a *no-sale interval*, i.e., an interval in which the firm does not sell the product, some of the customers whose reservation price is higher than the product's price are lost, some backlog, and the others buy early. The partition among these three groups in each no-sale interval depends on the good's price, the demand distribution, the customer's reservation price, holding and shortage cost parameters and the length of the no-sale interval. We further assume a complete-knowledge model in which the retailer knows the holding and shortage cost rates of customers, and the potential customers know its price and when the product is available for sale. We consider here two customer types each characterized by (1) its reservation price; and (2) its constant demand rate if the product were available continuously at a price lower than its reservation price. If the price is higher than the reservation price of this type of customers, then the respective demand rate drops to zero. Thus, in general, the demand rate declines with the product's price. If the product is not sold continuously, as permitted by our model, then customers whose reservation price is higher than the price of the product, and the time they most desire it falls within an interval in which it is not sold, may decide to buy it earlier, later or quit without buying it. In general, the demand increases as the time the product is available on the shelf increases. Under this general setting, our ultimate goal is to identify an optimal pricing and inventory replenishment policy that maximizes the profit rate of the retailer.

The literature commonly considers either pure backlogging or pure lost sales in no-sale intervals; see for example Section 3.3 in Zipkin (2000). Under backlogging, which holds especially for monopolists, customers that arrive in no-sale intervals wait for the next reordering time. The retailer is then penalized by a shortage cost consisting of the administrative work involved in handling the shortage and the loss of good will. The shortage cost is usually assumed to be proportional to the amount backlogged and to the backlogging duration; see Veinott (1966). Rosling (2002) considers two non-linear backorder costs; a fixed cost per unit (independent of the backorder duration) (see also Hadley and Whitin (1963) and Chen and Zheng (1993)), and a cost proportional to the time backlogs stay on the book, regardless of the quantity backlogged (see also Silver and Peterson (1979)). In the lost sales case, demands that occur when no inventory is available are lost forever. The unit penalty cost in this case includes also the opportunity cost. The backlogging/lost sales cost should represent all negative consequences of the shortage. However, it is well known that it is extremely difficult for managers to assess these costs, and modelers thus resort both in practice and in theory to an alternative approach in which a service level constraint is imposed either on the fill rate or on the stockout frequency, see Zipkin (2000).

Pricing models assume in general that a customer buys the product if its reservation price is at least as high as its cost. Most of the existing pricing models that allow backlogging assume that customers whose reservation price is at least as large as the unit price wait patiently to the next reorder time. Partial backlogging is often assumed in the literature on perishable items (see for example, Abad (1996)), as then the customer may have an incentive to wait for a new replenishment to get a fresh product. We consider here a product whose quality does not deteriorate over time. A delay in the supply may thus cause in practice a monetary damage to the customer, and consequently the product may become less attractive for her and she may decide to drop out. Bucovetsky (1983) and Sobel (1984) do consider the shortage cost that customers pay in case of backlogging. Eppen and Libermann (1984) allow for customers to hold inventory. We follow Glazer and Hassin (1986,1990) in allowing customers to pay for either backlogging or for holding inventory. The assumption that customers are charged for shortages and for holding inventory induces a shortage cost for the retailer which does not necessitate the assessment of unit shortage/backlogging costs by the manager: the no-sale intervals shrink the market for the product. The higher are the holding/shortage costs faced by the customers, the higher is the shortage penalty cost faced by the retailer, due to loss of customers. Moreover, since the price of the product is also a decision variable in our model, the indirect shortage costs that the retailer faces are price-dependent. The assessment of the customer's inventory holding cost and her shortage cost is simpler than the assessment of the backlogging/lost sales cost of the firm as it does not involve the loss of good will cost. (Consider the example of the car vendor presented above.)

We review models that are closely related to ours. For an extensive review on coordinated pricing and production/procurement decisions, see Yano and Gilbert (2003). We start with the seminal paper by Whitin (1955) who considered the economic order quantity with pricing when the product must be sold continuously. In that paper the author assumed that the demand rate is a decreasing linear function of the price. The objective function in his model is the maximization of the profit rate, which is the product of the price by the respective demand rate minus the average-time cost which is calculated as in the EOQ formula. Our model builds directly on Glazer and Hassin (1986,1990). These two last papers assume as in Whitin (1955), a pricing and replenishment model with the following generalizations/modifications: (i) the retailer is now free to decide when to sell the product; (ii) customers are allowed, if they like, to buy the product earlier or later than the time they most need it paying either inventory holding or shortage cost; (iii) the reservation price of all customers is identical, so customers differ only in the time they most desire the product. Glazer and Hassin (1990) consider the same model as in Glazer and Hassin (1986) but the objective function is the maximization of the aggregate welfare of seller and customers. The authors show that the optimal policy is of one of the three types of policies as described in Glazer and Hassin (1986) and provide conditions and specifications under which each type is optimal.

Finally, we mention the extensive literature on inventory models with variable capacity, for example, Parlar and Berkin (1991), Wang and Gerchak (1996), Güllü (1998), and Güllü, Önol and Erkip (1999). In these models there are periods when supply is not available or is partially available. Customers are usually uninformed about when these "dry periods" start and end. Closest to our model is the model of Atasoy, Güllü and Tan (2010) that considers a discrete time three-level supply chain where a manufacturer orders supply from an external supplier that may stop selling in certain periods. However, in order to help the manufacturer, the supplier provides him an accurate information about the availability of the supply in the next given number of periods. The manufacturer is facing deterministic periodic demands of customers. The paper considers the manufacturer's problem who needs to plan its own order quantities from the supplier in order to minimize his total expected costs that consist of his ordering costs, plus holding and backorder costs, given the available limited information about the dry periods. In this model, and in ours, the supply is not available at all times, and the manufacturer (in their model) or the customers (in ours) need to decide when and if to buy in order to minimize their costs. Though, there are several differences between this model and ours, and the most crucial one is that in their model the timing and length of supply availability is not strategically planned but result from external random forces,

Let w_i be the reservation cost of customers of type i, $i = 1, 2$, which are also called *i-customers*, where $w_1 > w_2$. We identify nine possibilities for an optimal solution. Three possibilities consist of sales at replenishment instants only to all 1-customers, and to a proportion x of 2-customers, where $x = 0$, $x = 1$ or $0 < x < 1$. Two possibilities of continuous sales to either only 1-customers, or to all customers. We call such policies that consist of sales at replenishment instances or continuous sales policies - *simple policies.* Three more possibilities consist of a continuous sales interval followed by a no-sale interval. We call such policies *semi-continuous policies.* In the three possible optimal semi-continuous policies, sales are made to all 1-customers, and, in addition, to a proportion x of 2-customers appearing in the continuous sales interval, and to a proportion y of the 2-customers appearing in the no-sale interval, where $x = y = 0$, or $x = 1$ with $y = 0$ or $0 < y < 1$. Finally, we identify another possible optimal policy that consists of a continuous sales interval followed by at least two no-sales intervals. The sales at the continuous sales interval are to all customers, and at the no-sales intervals to 1-customers only. We derive the expected rate of profits in each case so that for any set of parameters the optimal solution is easily computed. We also give conditions that distinguish among possible optimal strategies. We expect that the insight gained by the analysis of this restricted case will help to shed light on the optimal pricing and replenishment policy for a general customers' distribution.

The paper is organized as follows: In Section 2 we describe the model and present notation with preliminary results. In Section 3 we consider simple policies of selling either only at replenishment instants or continuously through the cycle. In Section 4 we consider semi-continuous policies, where sales are made continuously through the first part of the cycle, followed by a no-stock interval. In Sections 5-8 we characterize the possible solutions which are neither simple nor semi-continuous depending on whether the customers' holding cost is greater or smaller than their shortage cost. For each policy type we compute the average rate of profit, V, of the policy of this type which can be a candidate for being optimal if certain conditions on the input parameters are satisfied. Altogether we identify nine types of policies. We number their solution values as $V^{(1)}, \ldots, V^{(9)}$. The optimal solution for a given set of input parameters is then the policy with highest value among those whose necessary conditions are satisfied, provided that this value is positive. The results are summarized in the concluding section.

2 The model and preliminaries

We consider a deterministic model with a monopolistic firm that sells a single type of a product at a constant price to two types of customers, i.e., no price discrimination is allowed. The firm incurs a fixed replenishment cost of size A, and a variable cost c. It also pays a linear holding cost of h_f per unit of the product per unit of time. Customers differ by two parameters, the time when they most need the product, which we call their *demand time*, and their reservation price, which is their valuation of the product at that point of time. If the product were free and it were available on the shelf continuously then the arrival rate of customers would have been constant with parameter λ. We assume that the first type of customers is characterized by a reservation price w_1, and by an arrival rate λ_1. The second type of customers is characterized by a reservation price w_2, and by an arrival rate λ_2, where $w_1 > w_2$ and $\lambda = \lambda_1 + \lambda_2$. We assume that each customer needs a single unit of the product. The following terminology is used: customers that demand the product at t are called *t-customers* and customers whose reservation price is w_i are called *i-customers*. The t-customers that are also i-customers are called (t, w_i)-customers. I.e., any (t, w_i)-customer demands the product at t and her reservation price is w_i.

Customers are assumed to be ready to buy the product earlier or later than her demand time. Such a deviation comes at a cost. Specifically, a (t, w)-customer is ready to pay for it at most $w - h_c\tau$ at time $t - \tau$ and at most $w - s\tau$ at time $t + \tau$ for any $\tau > 0$. This behavior can be interpreted as follows: By obtaining the product before t, the customer incurs an "inventory holding cost" of h_c per unit of time; by obtaining the product after t, the customer incurs "shortage cost" of s per unit of time. Clearly, a customer whose reservation price of the product upon her demand-time is less than the price of the product, will never buy the product. Thus, if the product were on the shelf continuously, the arrival rate of customers that are willing to pay at least p upon their demand-time would be λ if $p \leq w_2$, and would be λ_1 if $w_2 < p \leq w_1$. If $p > w_1$ there will be no market for the product. Clearly, an optimal price p should satisfy $p > c$, and as we deal here with two types of customers we assume that $c < w_2$.

Remark 1 If any of h_f, h_c, s is equal to 0 then the problem is trivial. Therefore, we assume that these parameters have strictly positive values.

The firm wants to maximize its average rate of profits by choosing a price p and a replenishment policy. Like in the EOQ model, there exists an optimal cyclic stationary policy, and without loss of generality we focus on the first cycle $[0, T]$. The ZIO (Zero-Inventory-Ordering) property holds here and therefore a new order is placed after the stock is depleted. However, unlike in the EOQ model, the firm is allowed not to sell the product continuously during the cycle if this increases its average rate of profit. We assume that the policy of the firm, namely the price and sale epochs, is known to the customers and therefore, a customer whose demand-time is at a no-sale point may decide to buy the product earlier or later when it is sold, or alternatively, to quit and not buy it at all. As a result, the optimal policy structure may be such that the stock depletes earlier than at T.

In this paper we fully characterize an optimal cyclic solution of our model for two types of customers. We show that there exists an optimal solution where for some $T_I \in [0, T]$, the firm sells continuously up to T_I. If the stock is depleted at T_I, then the interval (T_I, T) is a *no-stock interval*, i.e. an interval in which the firm does not hold any stock. In such a case, $T_I = 0$ means that the firm never holds stock and it sells only at replenishment epochs, and $T_I = T$ means that the firm sells continuously through the cycle . We call these two types of extreme policies *simple policies*:

Definition 2 A policy is *simple* if the sales are continuous or only at replenishment instants.

A semi-continuous policy is obtained if $0 < T_I < T$ and the stock is depleted at T_I, meaning that (T_I, T) is a no-stock interval:

Definition 3 A policy is *semi-continuous-sales*, or for short semi-continuous, if sales are continuous until the stock is depleted.

The interval $(a, a + \Delta)$ is said to be a *no-sale interval* if the firm sells the product only at points a and $a + \Delta$, and nowhere else in the interval. In particular, the no-stock interval (T_I, T) in simple and semi-continuous policies is a no-sale interval. Except of simple and semi-continuous policies defined above, other possible candidates for optimal cyclic policies exist. Such policies consist of an interval $[0, T_I]$ of continuous sales, $0 \leq T_I < T$, and thereafter the stock at T_I is sold at a number of discrete points before T. In other words, such a policy consists of a (possibly empty) interval $[0, T_I]$ of continuous sales, followed by at least two no-sales intervals, where the last one that ends at T is also a no-stock interval.

We distinguish several cases and solve each case separately. The optimal policy is obtained by solving all cases and picking up the best one. We use the following notation:

$\sigma = \frac{h_c s}{h_c + s}$

$\beta = \frac{s}{h_c + s}$,

$H_f = h_f A$,

$\sigma_A = \sigma A$,

and we denote the average cost per unit of time of a given policy by V. A policy is *profitable* if its average rate of profit is positive. If no profitable policy exists, it is optimal for the firm to do nothing. In such a case the optimal average profit is 0. As mentioned in the introduction, we compute nine candidate policies and mark their values as $V^{(1)}, \ldots, V^{(9)}$. *These values* *depend on the two input parameters σ_A and H_f, rather than on σ, h_f, and A.* It turns out that β plays a central role in the analysis. If sales are given at the ends of an interval but not within it, β is the fraction of the customers arriving during the interval and buying the product, who prefer advancing their purchase to its beginning, while the other fraction of $1-\beta$ defer their purchase to the end of the interval. Note that some of the customers arriving during such an interval may quit without buying the product, however, as we prove, it is never optimal for a profitable policy to contain a sub-interval such that all customers born in it are lost.

Lemma 4 *A profitable optimal policy does not contain a time interval such that all customers born in it are lost.*

Proof: Suppose that the there exists an optimal cyclic profitable policy Π, with price p and cycle $[0, T)$ that contains sub-intervals of total length $0 < \tau < T$ in which all customers are lost. Let $V(\Pi) > 0$ denote the average profit of Π. Consider an alternative policy Π', with a cycle length of $T - \tau$, that is exactly as Π except that all intervals in which all customers are lost are removed from the cycle. The number of customers in a cycle that buy the product in Π and Π' is exactly the same, and thus the revenue per cycle and the fixed cost per cycle are not affected by this change. Moreover, the holding cost per cycle of Π' is bounded from above by the holding cost per cycle of Π. Thus, the total profit in a cycle in Π' is at least as large as that of Π, and as the cycle length of Π' is smaller than that of Π, Π' is a strictly better policy, contradicting the optimality of Π. ■

As we show, it is most common that the optimal policy is either simple or semi-continuous, i.e., it is sub-optimal to insert no-sale intervals between T_I and the no-stock interval. However, for particular sets of input data this is possible.

We first analyze the behavior of customers who are arriving at a no-sale interval $(a, a + \Delta)$. For a fixed price p, define

$$w(p, \Delta) = p + \sigma\Delta, \tag{1}$$

and

$$\theta_{a,\Delta} = a + \beta\Delta.$$

Figure 1 illustrates $\theta_{a,\Delta}$ and $w(p, \Delta)$, and a fixed value of w such that $p < w < w(p, \Delta)$. The indifference curve of a (t, w)-customer describes how much such a customer is willing to pay for the product at every instance τ. It raises with slope $\frac{\sigma}{\beta}$ for $\tau < t$ and decreases with slope $\frac{\sigma}{(1-\beta)}$ for $\tau > t$. Given a no-sale interval $(a, a+\Delta)$, sales to (t,w)-customers with $a + \frac{\beta(w-p)}{\sigma} < t < a + \Delta - \frac{(1-\beta)(w-p)}{\sigma}$ are lost, (t,w)-customers with $a \leq t \leq \min\{a + \frac{\beta(w-p)}{\sigma}, \theta_{a,\Delta}\}$ buy at a, and those with $\max\{a+\Delta - \frac{(1-\beta)(w-p)}{\sigma}, \theta_{a,\Delta}\} \leq t \leq a+\Delta$ buy at $a+\Delta$. Observe that for a reservation price $w = w(p,\Delta)$, there is no loss of $w(p,\Delta)$-customers in a no-sale interval $(a, a+\Delta)$. Moreover, since $w(p,\Delta) - \frac{\sigma}{\beta}\beta\Delta = w(p,\Delta) - \frac{\sigma}{(1-\beta)}(1-\beta)\Delta = p$, a $(\theta_{a,\Delta}, w(p,\Delta))$-customer is indifferent among buying at a, buying at $a+\Delta$, and not buying at all. If $\beta > 0.5$ then the sales at a are higher than at $a+\Delta$, and when $\beta \leq 0.5$, the sales at $a+\Delta$ are higher than at a.

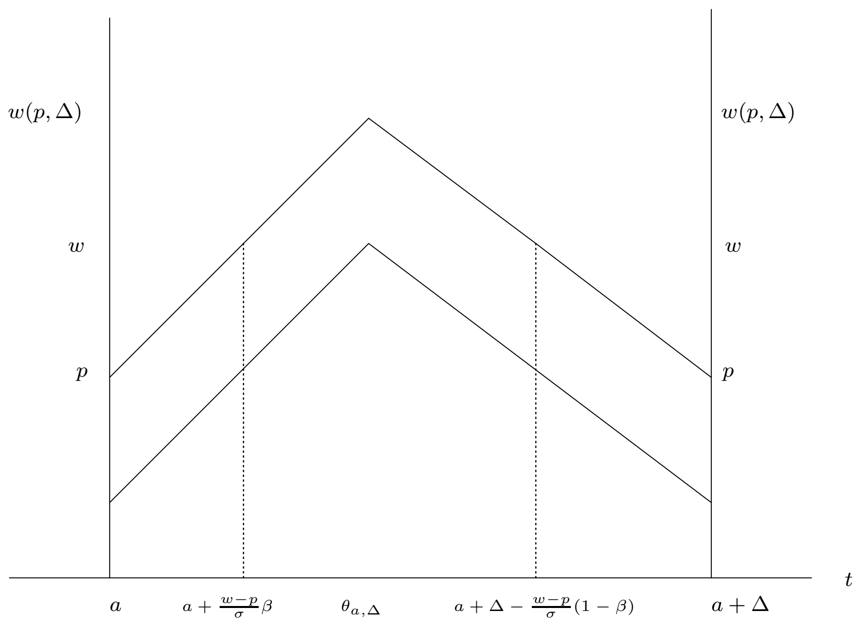

Figure 1: Indifference curves of a $(\theta_{a,\Delta}, w)$-customer and a $(\theta_{a,\Delta}, w(p,\Delta))$-customer

From now on we focus on 2 customer types. Let $\Delta_i = \frac{w_i - p}{\sigma}$ denote the maximum length of a no-sale interval without loss of any $i - customers$. In the following we compute the sales volume at the two extreme points of a no-sale interval $(\tau_0, \tau_0 + x)$ from those customers arriving in the interval.

The sales volume at τ_0 from those customers is

$$\lambda_1 \beta \min(x, \Delta_1) + \lambda_2 \beta \min(x, \Delta_2),$$

The sales volume at $\tau_0 + x$ from those customers is

$$\lambda_1(1-\beta) \min(x, \Delta_1) + \lambda_2(1-\beta) \min(x, \Delta_2).$$

3 Simple solutions

Each of the two kinds of simple policies is considered separately.

3.1 Sales at replenishment instants

There are four cases with sales made only at replenishment instants that need to be considered. Observe that in this case $T \in [\Delta_2, \Delta_1]$, because if $T > \Delta_1$ there are lost 1-customers which is impossible by Lemma 4. and if $T < \Delta_2$ then p can be increased without loosing sales.

- Sales only to 1-customers

  This may happen if the seller chooses $p \geq w_2$. In this case, $T = \Delta_1 = \frac{w_1 - p}{\sigma}$, or $p = w_1 - \sigma T$. Thus, $V(T) = \lambda_1[(w_1 - c) - \sigma T] - \frac{A}{T}$. It is optimized at $T = \sqrt{\frac{A}{\lambda_1 \sigma}}$. If $\sqrt{\frac{\sigma_A}{\lambda_1}} > w_1 - w_2$, then $p < w_2$, implying that this solution can be ignored as selling to the two types of customers at the replenishment epochs gives a better solution. The resulting solution is given in Figure 2.

Let $p = w_1 - \sqrt{\frac{\sigma_A}{\lambda_1}}$;
If $p \geq w_2$ then
$T = \sqrt{\frac{A}{\lambda_1 \sigma}}$;
$V^{(1)} = \lambda_1(w_1 - c) - 2\sqrt{\lambda_1 \sigma_A}$.

Figure 2: Sales to 1-customers at replenishment instants

We next proceed to the cases of sales to both types of customers, i.e. $p < w_2$.

- No loss of customers

  There is no loss of customers if $p < w_2$ and $T \leq \Delta_2$, but clearly $T < \Delta_2$ is suboptimal since in this case p can be increased without loosing sales. Hence, $T = \Delta_2$, and $V(T) = \lambda[(w_2 - c) - \sigma T] - \frac{A}{T}$ is optimized at the solution given in Figure 3.

$$\boxed{\begin{array}{l} p = w_2 - \sqrt{\frac{\sigma_A}{\lambda}}; \\ \text{If } p > c \text{ then} \\ T = \frac{1}{\sigma}\sqrt{\frac{\sigma_A}{\lambda}}; \\ V^{(2)} = \lambda(w_2 - c) - 2\sqrt{\lambda\sigma_A}. \end{array}}$$

Figure 3: Sales to all customers at replenishment instants

- Sales to all 1-customers and some 2-customers with $T = \Delta_1$

  Here $T = \Delta_1 = \frac{w_1 - p}{\sigma}$ or $p = w_1 - \sigma T$, giving $\Delta_2 = \frac{w_2 - p}{\sigma} = T - \frac{w_1 - w_2}{\sigma}$. Hence,

$$\begin{aligned} V(T) &= \frac{1}{T}\left\{[\lambda_1 T + \lambda_2\Delta_2][(w_1 - c) - \sigma T] - A\right\} \\ &= -\lambda\sigma T + [\lambda(w_1 - c) + \lambda_2(w_1 - w_2)] - \frac{1}{T}\left(\lambda_2(w_1 - c)\frac{w_1 - w_2}{\sigma} + A\right) \\ &= \alpha_1 T + \alpha_0 + \frac{\alpha_{-1}}{T}, \end{aligned}$$

  where $\alpha_{-1} = -\left(\lambda_2(w_1 - c)\frac{w_1 - w_2}{\sigma} + A\right)$, $\alpha_0 = \lambda(w_1 - c) + \lambda_2(w_1 - w_2)$, and $\alpha_1 = -\lambda\sigma$. The optimal solution has $T = \sqrt{\frac{\alpha_{-1}}{\alpha_1}}$ and the profit rate is $\alpha_0 - 2\sqrt{\alpha_{-1}\alpha_1}$, as both α_{-1} and α_1 are negative. This solution is feasible if $p < w_2$, or equivalently $\lambda(w_1 - w_2)^2 < \lambda_2(w_1 - c)(w_1 - w_2) + \sigma_A$. The solution is given in Figure 4.

- $\Delta_2 < T < \Delta_1$

  The analysis is as in the previous case but here we also need to compute the optimal p:

  $$V(T, p) = \frac{1}{T}\left\{(\lambda_1 T + \lambda_2\Delta_2)(p - c) - A\right\} = \frac{1}{T}\left\{\left(\lambda_1 T + \lambda_2\frac{w_2 - p}{\sigma}\right)(p - c) - A\right\}.$$

$p = w_1 - \sqrt{\frac{\lambda_2(w_1-c)(w_1-w_2)+\sigma_A}{\lambda}}$;
If $c < p < w_2$ then
$T = \frac{w_1-p}{\sigma}$;
$V^{(3)} = \lambda(w_1 - c) + \lambda_2(w_1 - w_2) - 2\sqrt{\lambda[\sigma_A + \lambda_2(w_1 - c)(w_1 - w_2)]}$.

Figure 4: Sales to all 1-customers and some 2-customers at replenishment instants

Equating to 0 the partial derivative with respect to p gives the optimal value $p = \frac{1}{2}\left(\frac{\lambda_1}{\lambda_2}\sigma T + w_2 + c\right)$ and $\Delta_2 = \frac{1}{2}\left(\frac{w_2-c}{\sigma} - \frac{\lambda_1}{\lambda_2}T\right)$. Recall that in this case $p < w_2$, which is equivalent to $T < \frac{\lambda_2}{\lambda_1}\frac{w_2-c}{\sigma}$. In this range $V(T) = \alpha_1 T + \alpha_0 + \frac{\alpha_{-1}}{T}$, where $\alpha_1 = \frac{\lambda_1^2\sigma}{4\lambda_2}$, $\alpha_0 = 0.5\lambda_1(w_2 - c)$, and $\alpha_{-1} = \frac{\lambda_2(w_2-c)^2}{4\sigma} - A$.

Let $\gamma = \frac{\lambda_2}{\lambda_1}\frac{w_2-c}{\sigma}$. $T = \gamma$ means that $p = w_2$ with sales only at replenishment points, i.e., selling to 1-customers only, which we have already considered. Thus, assume $T < \gamma$. If $\alpha_{-1} \leq 0$ then $V'(T) > 0$ and thus $T = \min\{\Delta_1, \gamma\}$. If $\alpha_{-1} > 0$ then the function $V(T)$ is convex and the maximum is obtained at an extreme point $T \in \{\Delta_2, \min\{\Delta_1, \gamma\}\}$. I.e., in both cases there is no internal maximum of $V(T)$, so that $V(T)$ is maximized at a boundary point whose cost is derived in one of the other cases.

3.2 Continuous sales

We observe that continuous sales with price $w_2 < p < w_1$ is never optimal because 2-customers don't buy and an increase in p doesn't cause loss of 1-customers.

- $p = w_1$: This is the maximum value that the price can assume. An optimal policy with $p = w_1$ must be a continuous-sales policy. In such a case only 1-customers buy and $V(T) = \lambda_1(w_1 - c) - 0.5h_f\lambda_1 T - \frac{A}{T}$ is optimized at

$$T = \sqrt{\frac{2A}{\lambda_1 h_f}},$$

giving the solution described in Figure 5.

$$p = w_1;$$
$$T = \sqrt{\frac{2A}{\lambda_1 h_f}};$$
$$V^{(4)} = \lambda_1(w_1 - c) - \sqrt{2\lambda_1 H_f}.$$

Figure 5: Continuous sales to 1-customers

- $p = w_2$: Here both customer types buy the product and $V(T) = \lambda(w_2 - c) - 0.5h_f\lambda T - \frac{A}{T}$ implying that $T = \sqrt{\frac{2A}{\lambda h_f}}$. The solution is given in Figure 6.

$$p = w_2;$$
$$T = \sqrt{\frac{2A}{\lambda h_f}};$$
$$V^{(5)} = \lambda(w_2 - c) - \sqrt{2\lambda H_f}.$$

Figure 6: Continuous sales to all customers

4 Semi-continuous solutions

In this section we characterize cases in which the optimal solution is semi-continuous. We deal separately with three cases according to the position of p relative to the interval $[w_2, w_1)$. We note that $p = w_1$ implies the continuous sales policy to 1-customers, as in Figure (5), and if $p > w_1$ no customers buy the product. Thus it is sufficient to consider $p < w_1$.

4.1 $p < w_2$

Note that in an optimal solution $T - T_I \in [\Delta_2, \Delta_1]$, since a longer no-stock interval causes loss of 1-customers and a shorter one means that it is possible to raise p without affecting the sales pattern. The rate of profit is

$$V(T,T_I,p) = \frac{1}{T}\left\{(p-c)[\lambda_1 T + \lambda_2(T_I+\Delta_2)] - h_f\left[0.5\lambda T_I^2 + \beta T_I[\lambda_1(T-T_I)+\lambda_2\Delta_2]\right] - A\right\}.$$

Looking for an internal solution with respect to p, we equate to zero the partial derivative with respect to p, giving

$$p = \frac{1}{2}\left\{\left(\frac{\lambda_1}{\lambda_2}T + T_I\right)\sigma + (w_2+c) + h_f\beta T_I\right\},$$

and therefore, $\Delta_2 = \frac{1}{2\sigma}[(w_2 - c - \frac{\lambda_1}{\lambda_2}T\sigma) - (\sigma + h_f\beta)T_I]$. These relations imply that $V(T,T_I) = \frac{1}{T}\left[a\sigma T_I^2 + b(T)T_I + d(T)\right]$, where

$$\begin{aligned}
a &= \frac{1}{4}\lambda_2 + \frac{1}{2\sigma}h_f\beta\lambda_2 + \frac{1}{4\sigma^2}h_f^2\beta^2\lambda_2 - \frac{1}{2\sigma}h_f\lambda + \frac{1}{\sigma}h_f\lambda_1\beta \\
&= \frac{\lambda_2}{4\sigma^2}(\sigma+h_f\beta)^2 - \frac{h_f}{\sigma}\left(\frac{\lambda}{2} - \lambda_1\beta\right), \\
b(T) &= \frac{1}{2}(\sigma - h_f\beta)\left(\lambda_1 T + \frac{\lambda_2(w_2-c)}{\sigma}\right), \\
d(T) &= \frac{\sigma}{4\lambda_2}\left(\lambda_1 T + \frac{\lambda_2(w_2-c)}{\sigma}\right)^2 - A.
\end{aligned}$$

Equating to zero the partial derivative of $V(T,T_I)$ with respect to T_I gives $T_I = -\frac{b(T)}{2a\sigma}$ and for an internal solution we need $ab(T) < 0$, $T - T_I \in [\Delta_2, \Delta_1]$, and $c < p < w_2$.

Substituting T_I into $V(T,T_I)$ gives

$$V(T) = \frac{1}{T}\left(-\frac{b^2(T)}{4a\sigma} + d(T)\right) = \alpha_1\sigma T + \alpha_0 + \frac{\alpha_{-1}}{T}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{\lambda_1^2}{4}\left(-\frac{(\sigma-h_f\beta)^2}{4a\sigma^2} + \frac{1}{\lambda_2}\right), \\
\alpha_0 &= 2\alpha_1\lambda_2\frac{w_2-c}{\lambda_1}, \\
\alpha_{-1} &= \alpha_1\sigma\left(\frac{\lambda_2(w_2-c)}{\lambda_1\sigma}\right)^2 - A.
\end{aligned}$$

The sign of $b(T)$ is determined by the sign of $(\sigma - h_f\beta) = \beta(h_c - h_f)$. We consider three cases: $h_c = h_f$; $h_c > h_f$; and $h_c < h_f$.

- $h_c = h_f$: In this case $b(T) = T_I = 0$, implying that an optimal semi-continuous sales policy does not exist.

- $h_c > h_f$: In this case $b(T) > 0$, and we must have $a < 0$ to have an internal optimal solution. This also implies $\alpha_1 > 0$. Now, $T_I = -\frac{b(T)}{2a\sigma}$ and p are linear increasing functions of T, while Δ_2 and Δ_1 are linear decreasing functions of T.

  $V(T)$ is convex if $\alpha_{-1} \geq 0$ and otherwise it is concave. If it is convex, its maximum is obtained at an extreme point. In this case either T is as large as possible, namely the T that gives $p = w_2$, which is not the case considered here, or T is as small as possible, i.e., when either $T = T_I$ or $T_I = 0$, resulting in a simple solution considered in Section 3. If $\alpha_{-1} < 0$ then $V(T)$ is monotone increasing and concave, meaning again that there is no internal solution.

- $h_c < h_f$: In this case $b(T) < 0$. For an internal solution for T_I to exist, we must have $a > 0$. Also in this case T_I and p are linear increasing functions of T, while Δ_2 is linear decreasing in T.

  - If $\alpha_{-1} \geq 0$, $V(T)$ is convex and therefore, the optimal T is at an extreme value and either $p = w_2$, which is not the case considered here, or $T \in \{0, T_I\}$, which gives a simple solution.
  - If $\alpha_{-1} < 0$, $V(T)$ is concave.
    - If $\alpha_1 \geq 0$, $V(T)$ is an increasing function, attaining its maximum at the extreme value where $p = w_2$.
    - If $\alpha_1 < 0$ then the maximum of $V(T)$ is obtained at a value that satisfies $T = \sqrt{\frac{\alpha_{-1}}{\alpha_1 \sigma}}$. This case requires further investigation. Note that $\alpha_1 < 0$ is equivalent to $0 < a < \frac{\lambda_2(\sigma - h_f\beta)^2}{4\sigma^2}$. Substituting a in this inequality, $\alpha_1 < 0$ is equivalent to $\beta < 0.5$. Thus, $\beta < 0.5$ implies that both $\alpha_1 < 0$ and $\alpha_{-1} < 0$.

  This internal semi-continuous solution is given in Figure 7.

4.2 $p = w_2$

In this case both customer types buy the product during $[0, T_I]$, and only 1-customers buy it during (T_I, T). As we consider here semi-continuous

If $\beta < 0.5$, $h_c < h_f$, and if $a > 0$, $c < p < w_2$, and $T - T_I \in [\frac{w_2 - p}{\sigma}, \frac{w_1 - p}{\sigma}]$, where
$a = \frac{\lambda_2}{4}\left(1 + \frac{H_f}{\sigma_A}\beta\right)^2 - \frac{H_f}{\sigma_A}\left(\frac{\lambda}{2} - \lambda_1\beta\right)$;
$\alpha_1 = \frac{\lambda_1^2}{4}\left(-\frac{1}{4a}\left(1 - \frac{H_f}{\sigma_A}\beta\right)^2 + \frac{1}{\lambda_2}\right)$;
$T = \sqrt{\left(\frac{\lambda_2(w_2-c)}{\lambda_1\sigma}\right)^2 - \frac{A}{\alpha_1\sigma}}$;
$T_I = -\frac{1}{4a}(\sigma - h_f\beta)\left(\lambda_1 T + \lambda_2\frac{w_2-c}{\sigma}\right)$;
$p = \frac{1}{2}\left\{\left(\frac{\lambda_1}{\lambda_2}T + T_I\right)\sigma + (w_2 + c) + h_f\beta T_I\right\}$;
then,
$V^{(6)} = 2\alpha_1\left(\lambda_2\frac{w_2-c}{\lambda_1} + \sqrt{\left(\lambda_2\frac{w_2-c}{\lambda_1}\right)^2 - \frac{\sigma_A}{\alpha_1}}\right)$.

Figure 7: Semi-continuous sales: $p < w_2$

policies, we restrict ourselves to $0 < T_I < T$. Clearly, $T - T_I \le \Delta_1$ in order to avoid loss of 1-customers.

$$\begin{aligned} V(T, T_I) &= \lambda_1(w_2 - c) + \frac{1}{T}\left\{\lambda_2(w_2 - c)T_I - h_f\left[\frac{1}{2}\lambda T_I^2 + T_I\beta\lambda_1(T - T_I)\right] - A\right\} \\ &= \lambda_1(w_2 - c) + \frac{1}{T}\left\{T_I^2 h_f\left(\beta\lambda_1 - \frac{1}{2}\lambda\right) + T_I(\lambda_2(w_2 - c) - h_f\lambda_1\beta T) - A\right\}. \end{aligned}$$

Fix T.

- If $2\beta\lambda_1 \le \lambda$, the function $V(T_I)$ is concave, implying a candidate for an internal maximum, namely $T_I = \frac{\lambda_2(w_2-c) - \beta\lambda_1 h_f T}{h_f(\lambda - 2\beta\lambda_1)}$. This candidate is relevant (internal) only if $0 < T_I < T$, and $T - T_I < \Delta_1$.

  Substituting T_I in $V(T, T_I)$ gives

$$\begin{aligned} V(T) &= \lambda_1(w_2 - c) + \frac{1}{T}\left\{\frac{2(\lambda_2(w_2 - c) - h_f\beta\lambda_1 T)^2}{(\lambda - 2\beta\lambda_1)h_f} - A\right\} \\ &= \lambda_1(w_2 - c) + \frac{2}{(\lambda - 2\beta\lambda_1)h_f}\left\{\alpha_1 T + \alpha_0 + \frac{\alpha_{-1}}{T}\right\}, \end{aligned}$$

where $\alpha_{1} = h_f^2\beta^2\lambda_1^2 > 0$, and $\alpha_{-1} = \lambda_2^2(w_2 - c)^2 - 0.5(\lambda - 2\beta\lambda_1)h_f A$. If $\alpha_{-1} \geq 0$ then $V(T)$ is convex, and if $\alpha_{-1} < 0$ then it is monotone increasing. In both cases the maximum is obtained at an extreme value of T where $T_I \in \{0, T, T - \Delta_1\}$. The solutions $T_I \in \{0, T\}$ are simple, and were considered in Section 3. Thus, only $T_I = T - \Delta_1$ is relevant.

- If $2\beta\lambda_1 > \lambda$, the function $V(T_I)$ is convex and its maximum is obtained at a boundary value, $T_I \in \{0, T, T - \Delta_1\}$. As only semi-continuous policies are considered here, we get also here that only $T_I = T - \Delta_1$ is relevant.

Therefore, we continue by substituting $T_I = T - \Delta_1$ into $V(T, T_I)$:

$$\begin{aligned} V(T) &= \lambda_1(w_2 - c) + \frac{1}{T}\left\{\lambda_2(w_2 - c)(T - \Delta_1) - h_f\left[\frac{\lambda}{2}(T - \Delta_1)^2 + (T - \Delta_1)\Delta_1\beta\lambda_1\right] - A\right\} \\ &\equiv \frac{\alpha_{-1}}{T} + \alpha_0 + \alpha_1 T, \end{aligned}$$

where $\alpha_{-1} = \Delta_1^2 h_f(\beta\lambda_1 - 0.5\lambda) - \Delta_1\lambda_2(w_2 - c) - A \leq 0$, $\alpha_0 = \lambda(w_2 - c) + h_f\Delta_1(\lambda - \beta\lambda_1)$, and $\alpha_1 = -0.5h_f\lambda < 0$. If $\alpha_{-1} = 0$, $V(T)$ is decreasing in T, and therefore the maximum is obtained at $T = \Delta_1$, resulting in a simple solution, see Section 3. If $\alpha_{-1} < 0$, then $V(T)$ is concave having an internal maximum with $T = \sqrt{\frac{\alpha_{-1}}{\alpha_1}}$ and profit $\alpha_0 - 2\sqrt{\alpha_1\alpha_{-1}}$, as described in Figure 8.

$p = w_2$;
$\Delta_1 = \frac{w_1 - w_2}{\sigma}$;
$T = \sqrt{\frac{\lambda_2(w_2 - c)\Delta_1 + 0.5h_f\Delta_1^2(\lambda - 2\beta\lambda_1) + A}{0.5h_f\lambda}}$;
$T_I = T - \Delta_1$;
If $T_I > 0$ then
$V^{(7)} = \lambda(w_2 - c) - H_f\frac{w_1 - w_2}{\sigma_A}(-\lambda + \beta\lambda_1) - \sqrt{2\lambda H_f\left[\frac{w_1 - w_2}{\sigma_A}\left(\lambda_2(w_2 - c) + H_f\frac{w_1 - w_2}{\sigma_A}\left(\frac{\lambda}{2} - \beta\lambda_1\right)\right) + 1\right]}$.

Figure 8: Semi-continuous sales: $p = w_2$

4.3 $w_2 < p < w_1$

Here only 1-customers buy. Again, as in this section we deal with semi-continuous policies, we look for solutions with $0 < T_I < T$. In addition, $T - T_I \leq \Delta_1$ in order to avoid loss of 1-customers in some intervals. Thus,

$$V(T, T_I, p) = \lambda_1(p - c) - \frac{1}{T}\left(0.5\lambda_1 h_f T_I^2 + \lambda_1 h_f \beta(T - T_I)T_I + A\right).$$

In an optimal solution $T - T_I = \Delta_1 = \frac{w_1 - p}{\sigma}$, otherwise p can be increased to increase profits. Substituting $p = w_1 - \sigma(T - T_I)$ into the cost function we get:

$$V(T, T_I) = \lambda_1\left(w_1 - c - \sigma(T - T_I)\right) - \frac{1}{T}\left(0.5\lambda_1 h_f T_I^2 + \lambda_1 h_f \beta(T - T_I)T_I + A\right).$$

By fixing T we get $V(T_I) = a(T) + bT_I + c(T)T_I^2$, where

$$a(T) = \lambda_1\left(w_1 - c - \sigma T\right) - \frac{A}{T},$$

$$b = \lambda_1(\sigma - h_f\beta) > 0,$$

$$d(T) = \frac{\lambda_1 h_f}{T}(\beta - 0.5).$$

The sign of $d(T)$ is determined by the sign of $\beta - 0.5$. If $d(T) \geq 0$ then $V(T_I)$ is convex in T_I, meaning that its maximum is obtained at an extreme value of T_I, namely $T_I \in \{0, T\}$, both are simple solutions that were considered in Section 3. Thus, suppose that $d(T) < 0$, or equivalently $\beta < 0.5$. In this case $V(T_I)$ is concave in T_I and a possible internal maximum is

$$T_I = -\frac{b}{2c(T)} = \frac{\sigma - h_f\beta}{h_f(1 - 2\beta)}T.$$

In order for T_I to be positive, and because $\beta < 0.5$, it must hold that $\sigma - h_f\beta > 0$, which is equivalent to $h_c > h_f$. In addition we need $T_I < T$, which holds only if $h_f > s$. The two conditions $h_f < h_c$ and $h_f > s$ imply that $\beta < 0.5$. Note also that the condition $p > w_2$ implies that $T - T_I < \frac{w_1 - w_2}{\sigma}$.

Substituting the expression for T_I into $V(T, T_I)$ gives

$$V(T) = \lambda_1(w_1 - c) - \frac{A}{T} + \frac{(\sigma + h_f\beta)^2 - 2\sigma h_f}{2h_f(1 - 2\beta)}\lambda_1 T,$$

which is a concave function of T. If the coefficient of T is nonnegative, $V(T)$ is increasing meaning that its maximum is obtained at an extreme point which occurs when $p = w_2$, a case which we don't consider here. Otherwise, if the coefficient of T is negative, (that is, $(\sigma + h_f\beta)^2 < 2\sigma h_f$) $V(T)$ is maximized at $T = \sqrt{\frac{2Ah_f}{\lambda_1}\frac{1-2\beta}{2\sigma h_f-(\sigma+h_f\beta)^2}}$. The resulting solution is given in Figure 9.

$T = \sqrt{\frac{2Ah_f(1-2\beta)}{\lambda_1\left[2\sigma h_f-(\sigma+h_f\beta)^2\right]}}$;
$T_I = \frac{(\sigma-h_f\beta)}{h_f(1-2\beta)}T$;
$p = w_1 - \sigma(T - T_I)$;
If $s < h_f < h_c$, $(\sigma + h_f\beta)^2 < 2\sigma h_f$, $p > w_2$, and $T - T_I \leq \frac{w_1-p}{\sigma}$ then
$V^{(8)} = \lambda_1(w_1 - c) - 2\sqrt{\frac{\lambda_1}{2H_f}\frac{2\sigma_A H_f-(\sigma_A+H_f\beta)^2}{1-2\beta}}$.

Figure 9: Semi-continuous sales: $w_2 < p < w_1$

5 Solutions which are neither simple nor semi-continuous

In the next lemma we prove that optimal policies which are neither simple nor semi-continuous consist of a single, possibly empty, continuous-sales interval that starts at the replenishment epoch, followed by at least two no-sales intervals. The next lemma provides some further properties of such an optimal solution.

Lemma 5 *There exists an optimal solution where the no-sale intervals are ordered in nondecreasing length. Moreover, a no-sale interval is not followed by an interval of continuous sales (hence there may be at most one continuous-sales interval and it must start at 0).*

Proof: We prove the first part of the theorem. The second part can be considered as a limit case and be proved similarly. Consider consecutive sales at $\tau_0, \tau_0 + x, \tau_0 + x + y$, and suppose that $x > y$. We will show that selling at $\tau_0 + y$ instead of at $\tau_0 + x$ does not decrease profits. W.l.o.g. let $\tau_0 = 0$ and $h_f = 1$. The change doesn't affect the total sales and therefore we only consider inventory holding costs. We assume in the analysis below that $x + y < T$, since if $x + y = T$ (i.e., $[x, x + y]$ is a no-stock interval) it is clear that the savings associated with selling earlier and thus postponing more sales to T are greater, so that the claim in this case also follows.

Let $C_i(x)$ ($C_i(y)$) be the holding cost associated with i-customers if the product is sold at x (y, respectively). We distinguish three cases:

- $x, y \leq \Delta_i$. In this case

  $$\begin{aligned} C_i(x) &= \lambda_i\{x[(1-\beta)x + \beta y] + [(x+y)y(1-\beta)]\} \\ &= \lambda_i[(x^2 + y^2)(1-\beta) + xy]. \end{aligned}$$

  and $C_i(x) = C_i(y)$.

- $y \leq \Delta_i < x$. In this case

  $$C_i(x) = \lambda_i\{x[(1-\beta)\Delta_i + \beta y] + [(x+y)y(1-\beta)]\},$$

  and
  $$C_i(y) = \lambda_i\{y[(1-\beta)y + \beta\Delta_i] + [(x+y)\Delta_i(1-\beta)]\},$$
  giving $C_i(x) - C_i(y) = \lambda_i y(x - \Delta_i) > 0$ by our assumption that $x > \Delta_i$.

- $y > \Delta_i$. Also here $C_i(x) > C_i(y)$. The sales to i-customers at $x + y$ are of size $(1-\beta)\Delta_i$, and at the middle sales point (x or y) they are of size Δ_i. But selling at y rather than at x saves in inventory costs.

∎

In view of the lemma, a general cyclic policy for the problem can be represented by a continuous-sales interval $[0, T_I]$, $T_I \geq 0$ followed by $k \geq 0$ no-sale intervals. In Sections 3 and 4 we considered the case $T_I = 0$ and $k = 1$, which is the simple policy with sales only at replenishment instants, the case $T_I = T$ and $k = 0$, which is the simple continuous-sales policy, and the case $0 < T_I < T$ and $k = 1$, which is the semi-continuous policy. Let x_i for $i = 1, \ldots, k$ denote the length of the $i-$th no-sale interval. Thus, $T = T_I + \sum_{i=1}^{k} x_i$. By Lemma 5, w.l.o.g. $x_1 \leq x_2 \leq \cdots \leq x_k$. Thus, the product is sold continuously in $[0, T_I]$, and thereafter positive quantities are sold in discrete points: $T_I + \sum_{i=1}^{\ell} x_i$ for $\ell = 0, 1, \ldots, k$.

6 High customer holding cost: $\beta \leq 0.5$

In this section we characterize the solution assuming $\beta \leq 0.5$, or equivalently, the customers' holding cost rate, h_c, is higher than their backlogging cost rate, s. We prove that in this case there exists an optimal solution which is either simple or semi-continuous. Then we analyze the case $\beta = 0.5$ separately as we use it in the next section to characterize the optimal solutions when $\beta > 0.5$.

Theorem 6 *If $\beta \leq 0.5$, there exists an optimal policy which is either simple or semi-continuous.*

Proof: Consider an optimal policy such that $T_I > 0$ is the last point of time in the cycle where the firm holds stock. Suppose also that in the given policy there exists a no-sale interval $[a, a + \Delta]$, such that $0 \leq a < a + \Delta \leq T_I$. Denote by D_a and $D_{a+\Delta}$ the amounts sold at the respective ends of the no-sale interval. According to the analysis of Section 2 and because $\beta \leq 0.5$, $D_a \leq D_{a+\Delta}$. Thus, the average holding cost paid by the firm for these units would decrease if it sold continuously to each of these customers at time they were born. Moreover, continuous sales may cause reneging customers born during the interval to buy as well, resulting in more profits to the firm. ∎

Indeed, in view of Theorem 6, the optimal solution for $\beta \leq 0.5$ can be obtained by enumerating all the solutions $V^{(\ell)}$ for $\ell = 1, \ldots, 8$ and picking up $\max\{0, V^{(1)}, V^{(2)}, \ldots, V^{(8)}\}$.

6.1 $\beta = 0.5$

According to Theorem 6, the optimal solution for $\beta = 0.5$ is either simple or semi-continuous. We next strengthen this result for the case that the optimal price $p \neq w_2$.

Theorem 7 *If $\beta = 0.5$ and $p \neq w_2$ then the optimal solution is simple.*

Proof: In view of Theorem 6 the optimal solution for $\beta = 0.5$ is either simple or semi-continuous. By Lemma 4, a necessary condition for a semi-continuous policy to be optimal is $p < w_1$. It remains to show that if $p < w_1$, $\beta = 0.5$ and $p \neq w_2$, there does not exist an optimal semi-continuous policy. If $p < w_2$, the only possible optimal semi-continuous policy obtained requires

$\beta < 0.5$, see Figure 7. If $w_2 < p < w_1$, the only possible optimal semi-continuous policy obtained requires $s < h_c$ which is equivalent to $\beta < 0.5$, see Figure 9, concluding the proof. ∎

This result turns out to be useful in the sequel when analyzing the case $\beta > 0.5$.

7 High shortage cost: $\beta > 0.5$, and $p \notin \{w_1, w_2\}$

We now analyze the case where the customer's shortage cost is higher than the customer's holding cost, i.e., $\beta > 0.5$. Theorem 8 states that when $\beta > 0.5$ it is suboptimal to have intervals of continuous sales, unless $p \in \{w_2, w_1\}$.

Theorem 8 *If $\beta > 0.5$ and $p \notin \{w_2, w_1\}$ then the optimal solution does not contain continuous-sales intervals.*

Proof: First we exclude semi-continuous policies. By Lemma 4 semi-continuous policies may be optimal only for $p < w_1$. According to Figures 7-9, if $p \neq w_2$, there does not exist a semi-continuous optimal policy if $\beta > 0.5$. Moreover, from Figures 5 and 6, a simple policy with continuous sales cannot be optimal. ∎

Theorem 8 implies that if $\beta > 0.5$, the optimal price p satisfies $p < w_1$. In the next two theorems we prove that for $\beta > 0.5$ and $p \neq w_2$ the optimal policy is sales at replenishment instants only.

Theorem 9 *Consider an optimal solution* sol *for an instance I defined by $\beta > 0.5$, c, h_f, h_c, s and A, with monotone non-decreasing no-sale intervals of lengths $x_1 \leq \cdots \leq x_k$. Define the* associated instance *I' with $c' = c$, $h'_f = h_f$, $A' = A$, and $h'_c = s' = \frac{2h_c s}{h_c+s}$. Note that $\beta' = \frac{1}{2}$ and $\sigma' = \sigma$. Denote the profits for these instances by $V(\text{sol})$ and $V'(\text{sol})$. Then $V(\text{sol}) \leq V'(\text{sol})$.*

Proof: Denote by D_i the total sales to customers arriving during the ith no-sale interval. Clearly $D_1 \leq \cdots \leq D_k$, and the sequence $D_1, \ldots, D_k$ depends on σ but not on β. However, β does affect the timing of the sales in the sales points. The sales at time $T_I + x_1 + \cdots + x_l$ for $\ell = 1, \ldots, k-1$ are $(1-\beta)D_l + \beta D_{l+1}$. Since $D_l \leq D_{l+1}$, the coefficients of β are nonnegative in all of these cases. It follows that the holding cost with $\beta > \frac{1}{2}$ is at least as large as the holding cost of the firm with $\beta = \frac{1}{2}$ *and the same value of* σ. Since the total revenue is the same, the profit decreases with β, for $\beta > 0.5$, while σ is kept constant. Therefore $V(\text{sol}) \leq V'(\text{sol})$. ∎

Theorem 10 *The optimal solution for an instance with $\beta > 0.5$ and $p \neq w_2$ is sales at replenishment instants only.*

Proof: Consider an instance I with $\beta > 0.5$ and its associated instance I' as in Theorem 9. Consider any non-simple solution sol with monotone nondecreasing no-sale intervals as in Lemma 5. By Theorem 7, if $\beta = 0.5$ and $p \neq w_2$ then there is an optimal simple solution sol$'$ to I'. Then,

$$V(\mathrm{sol}) \leq V'(\mathrm{sol}) \leq V'(\mathrm{sol}') = V(\mathrm{sol}').$$

The first inequality follows from Theorem 9, the second by optimality of sol$'$ to I', and the equality since the value of a simple solution depends on σ but not on β. Therefore sol$'$ is a better solution for I than sol. According to Lemma 8 an optimal solution for I does not contain a continuous-sales interval, implying that the optimal policy for I is simple with sales only at replenishment instants. ■

Corollary 11 *When $\beta > \frac{1}{2}$ the optimal solution is the best among the simple solutions with discrete sales, see Figures 2-4, and the best solution obtained under the assumption $p = w_2$.*

8 High shortage cost: $\beta > 0.5$, and $p = w_2$

Lemma 12 *The optimal solution consists of an initial interval of length $T_I \geq 0$ with continuous sales to both types, and sales to 1-customers only after this. These sales consist of $k \geq 0$ no-sale intervals all of length Δ_1, and possibly another no-sale interval of length $\alpha\Delta_1$ with $0 \leq \alpha < 1$ which starts at T_I.*

Remark 13 This section assumes $p = w_2$, and therefore only 1-customers buy in no-sale intervals. In view of Lemma 4, all no-sale intervals have a length of at most Δ_1. Considering the first case in the proof of Lemma 5, which is the case relevant here as there is no loss of 1-customers, the profit is not affected by the order of the no-sale intervals, except for the last one which must be the longest one.

Proof: By Lemma 5 there exists an optimal solution with nondecreasing no-sale intervals. If the claim doesn't hold then there exists an index $1 < i \leq k$ such that $x_{i-1} \leq x_i < \Delta_1$. We claim that increasing x_i while decreasing x_{i-1} by the same amount increases the profit. First note that the change does not affect sales since only 1-customers are involved and all no-sale intervals remain bounded by Δ_1. For comparing holding costs assume w.l.o.g. that x_{i-1} starts at 0, and we mark $x = x_1$. Let $x_1 + x_2 = \tau$ and by assumption $x \leq \tau/2$. It is sufficient to show that the holding cost is increasing in x. The inventory holding costs associated with customers born in $[0, \tau]$, for $\tau < T$, in the given solution amount to

$$C_1(x) = \lambda_1 h_f \{x[(1-\beta)x + \beta(\tau - x)] + \tau(\tau - x)(1-\beta)\}.$$

If $\tau = T$ the last term in the curly brackets should be removed. The derivative with respect to x for $\tau < T$ is proportional to

$$2(1-2\beta)x + \tau[2\beta - 1] \geq 2(1-2\beta)\frac{\tau}{2} + (2\beta - 1)\tau = 0,$$

where the inequality follows since $\beta \geq 0.5$ and $x \leq \tau/2$. If $\tau = T$ the derivative with respect to x is proportional to $2x(1-2\beta) + \beta\tau \geq \tau(1 - 2\beta) + \beta\tau = \tau(1-b) \geq 0$ for the same reasons as above. That means that the holding cost is increasing in x. Performing a sequence of changes of this type we end up with a solution as claimed and its cost is not greater than that of the original solution. ■

Therefore, for $\beta > 0.5$ and $p = w_2$, if the optimal solution is not simple then it falls in one of the following two options:

- A semi-continuous solution with $T_I = T - \Delta_1$ and of profit $V^{(7)}$, see Figure 8.

- A solutions which is neither simple nor semi-continuous. Such a solution consists of a continuous-sales interval $[0, T_I]$, $0 \leq T_I < T$, followed by $k+1$ no-sale intervals. The first of these intervals, i.e., the one starting at T_I, may be empty, and in any case its length is strictly less than Δ_1. All the other k no-sale intervals are of length Δ_1. In the sequel of this section we consider this type of policies that are neither simple nor semi-continuous, namely, policies with $k + \lceil\alpha\rceil \geq 2$.

The following observation states that the possibility of $0 < \alpha < 1$ and $k = 0$ can be excluded:

Observation 14 *Consider a problem with $\beta > 0.5$. If there exists an optimal policy with $p = w_2$ whose cycle contains a single no-sale interval (which is also a no-stock interval), then its length is Δ_1.*

The proof follows directly from Figures 2 and 8.

8.1 $0 < \alpha < 1$ and $k \geq 1$

In this subsection we prove that the possibility of $0 < \alpha < 1$ can be excluded from consideration also when $k > 0$.

Using Remark 13, we assume without loss of generality that the no-sale interval of length $\alpha\Delta_1$ starts at T_I. Fix T and k, then $T_I + \alpha\Delta_1$ is also fixed at value $T - k\Delta_1$, and $\frac{d\alpha}{dT_I} = -\frac{1}{\Delta_1}$. The terms in the profit function that are affected by the choice of α are: The profit from sales to 2-customers, $\lambda_2(w_2 - c)T_I$; The holding costs on sales in $[0, T_I)$, $0.5h_f(\lambda_1 + \lambda_2)T_I^2$; the holding costs on sales at T_I, $h_f\lambda_1\beta\alpha\Delta_1 T_I$; and the holding costs on sales at $T_I + \alpha\Delta_1$, $h_f\lambda_1(1-\beta)\alpha\Delta_1(T - k\Delta_1)$. We use $\frac{d\alpha}{dT_I} = -\frac{1}{\Delta_1}$ to obtain that the derivative of the profit V with respect to T_I is proportional to $\lambda_2(w_2 - c) - h_f\left[T_I\lambda_2 + \alpha\lambda_1\Delta_1(2\beta - 1)\right]$.

The second derivative is proportional to $2\beta\lambda_1 - \lambda$.

- Suppose $2\beta\lambda_1 \geq \lambda$. In this case, for any given T and $k \geq 1$, the profit function V is a convex function of T_I and therefore it is maximized at one of the extreme values $\alpha = 0$, $\alpha = 1$.

- Suppose $2\beta\lambda_1 < \lambda$. In this case, for any given T and $k \geq 1$, the profit function V is a concave function of T_I, and therefore the there is another candidate which satisfies the first-order optimality conditions. We will show that this solution cannot be optimal.

  Equating the derivative of V with respect to T_I to 0 gives,

$$T_I = \frac{w_2 - c}{h_f} - \alpha\Delta_1\lambda_1\frac{2\beta - 1}{\lambda_2}. \tag{2}$$

  Thus,

$$T = T_I + (\alpha + k)\Delta_1 = \frac{w_2 - c}{h_f} + k\Delta_1 - \alpha\Delta_1\frac{2\beta\lambda_1 - \lambda}{\lambda_2}. \tag{3}$$

  For this solution, with at least two no-sale intervals, to be optimal, the cost of carrying inventory to $T_I + \alpha\Delta_1$ must be smaller than the profit from selling there:

$$T_I + \alpha\Delta_1 = \frac{w_2 - c}{h_f} - \alpha\Delta_1\frac{2\beta\lambda_1 - \lambda}{\lambda_2} \leq \frac{w_2 - c}{h_f},$$

  or equivalently $2\beta\lambda_1 \geq \lambda$, contradicting the assumption of the claim. Therefore, an improved solution can be obtained by canceling the sale at $T_I + \alpha\Delta_1$. The effect of this cancellation is that some customers who previously bought there will buy instead at T_I and by that save the firm inventory holding costs. Others who previously bought there will not buy at all, and by assumption this also increases the firm's profit.

8.2 $\alpha = 0$

Denote $V(T) = V_k(T)$ if $T = T_I + k\Delta_1$, where $k \geq 1$. Thus,

$$V_k(T) = \lambda(w_2 - c) - \frac{1}{T}\left\{\lambda_2(w_2 - c)k\Delta_1 + \frac{1}{2}h_f\lambda T_I^2 + \lambda_1 h_f\left[T_I(k-1+\beta)\Delta_1 + k\frac{k-1}{2}\Delta_1^2\right] + A\right\}$$

$$= \lambda(w_2 - c)\frac{1}{T}\left\{\lambda_2(w_2 - c)k\Delta_1 + \frac{1}{2}h_f\lambda(T - k\Delta_1)^2 + \lambda_1 h_f\Delta_1\left[(T - k\Delta_1)(k-1+\beta) + k\frac{k-1}{2}\Delta_1\right] + A\right\}.$$

Substituting $T_I = T - k\Delta_1$ gives $V_k(T) = E - \left(BT + \frac{C}{T}\right)$, where
$E = \lambda(w_2 - c) + (w_1 - w_2)\frac{H_f}{\sigma_A}[\lambda_2 k + \lambda_1(1-\beta)]$;
$B = \frac{1}{2}h_f\lambda$;
$C = A + k\Delta_1\left[\lambda_2(w_2 - c) + \frac{1}{2}h_f\Delta_1(\lambda_2 k - \lambda_1(2\beta - 1))\right]$;
$BC = \frac{1}{2}\lambda H_f\left(1 + k\frac{w_1-w_2}{\sigma_A}\left[\lambda_2 w_2 + \frac{w_1-w_2}{2}\frac{H_f}{\sigma_A}(\lambda_2 k - \lambda_1(2\beta-1))\right]\right)$.

If $C < 0$ then $V_k(T)$ is decreasing in T and obtains its maximum at the lower boundary, $T = k\Delta_1$ and $T_I = 0$, so that there are sales to 1-customers only. Using the results of Glazer and Hassin (1986) for a single type of customers, such a structure with $k > 1$ is not possible. Therefore, we assume that $C > 0$. In this case $V_k(T)$ is concave with maximum at $T = \sqrt{C/B}$. The solution is given in Figure 10. Note that $V_1^{(9)} = V^{(7)}$, i.e., the case $k = 1$ is the semi-continuous case of Figure 8.

9 Summary

The nine possibilities for an optimal solution are:

- Sales at replenishment instants to 1-customers only, with value $V^{(1)}$.
- Sales at replenishment instants without loss of any customers, with value $V^{(2)}$.
- Sales at replenishment instants to all 1-customers but only a fraction of the 2-customers, with value $V^{(3)}$.

Let
$E = \lambda(w_2 - c) + (w_1 - w_2)\frac{H_f}{\sigma_A}[\lambda_2 k + \lambda_1(1-\beta)]$;
$B = \frac{1}{2}h_f\lambda$;
$C = A + k\Delta_1\left[\lambda_2(w_2 - c) + \frac{1}{2}h_f\Delta_1(\lambda_2 k - \lambda_1(2\beta - 1))\right]$.
If $C > 0$, and $\sqrt{C/B} \geq k\Delta_1$, then
$V_k^{(9)} = E - 2\sqrt{BC}$.

Figure 10: Solution with multiple no-sale intervals

- Continuous sales to 1-customers only at price w_1 and value $V^{(4)}$.
- Continuous sales to all customers at price w_2, with value $V^{(5)}$.
- Continuous sales to all customers followed by a no-sale interval with sales to all 1-customers but only a fraction of the 2-customers. This is a semi-continuous policy at price lower than w_2, and value $V^{(6)}$.
- Continuous sales to all customers followed by a no-sale interval with sales to 1-customers only. This is a semi-continuous policy at price w_2, and value $V^{(7)}$.
- Continuous sales to 1-customers followed by a no-sale interval with sales to 1-customers only. This is a semi-continuous policy at price higher than w_2 but lower than w_1, and value $V^{(8)}$.
- Continuous sales to all customers at price w_2 during an initial interval followed by k no-stock intervals in which 2-customers are lost, with value $V_k^{(9)}$.

We note that with a single type of customers, Glazer and Hassin (1986) proved that the optimal solution is either simple or semi-continuous. Indeed, multiplicity of no-sale intervals in our generalized model comes with $p = w_2$, meaning that if $w_1 = w_2$ the solution requires continuous sales, without a no-sale interval

Our computational experience demonstrated that the case, which is not possible with homogeneous customers, where $V_k^{(9)}$ is optimal for $k \geq 2$,

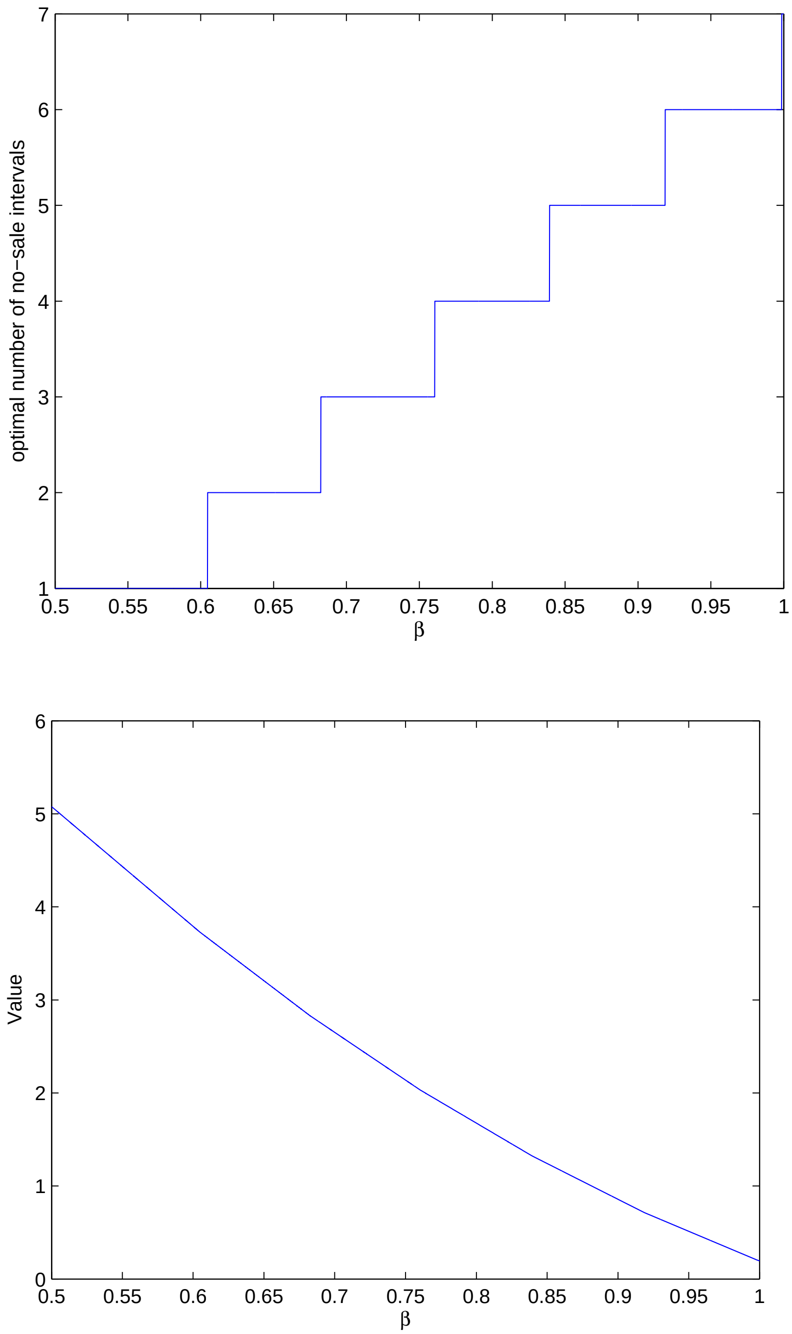

Figure 11: Optimal number of intervals (above) and the value function (below)

k	β	$V_1^{(9)}$	$V_2^{(9)}$	$V_3^{(9)}$	$V_4^{(9)}$	$V_5^{(9)}$	$V_6^{(9)}$	$V_7^{(9)}$
1	0.500	5.077	4.940	4.705	4.373	3.948	3.433	2.830
2	0.605	3.730	3.730	3.630	3.430	3.136	2.747	2.268
3	0.682	2.729	2.831	2.831	2.731	2.534	2.241	1.855
4	0.761	1.722	1.928	2.030	2.030	1.930	1.733	1.442
5	0.839	0.708	1.017	1.222	1.323	1.323	1.224	1.028
6	0.919	0	0.101	0.409	0.613	0.714	0.714	0.615
7	0.999	0	0	0	0	0.101	0.202	0.202

Table 1: The minimum $\beta \geq 0.5$ such that k no-sale intervals are optimal, and the values $V_i^{(9)}$ $i = 1, \ldots, 7$ at this β. $(w_1, w_2, l_1, l_2, H_f, \sigma_A, c) = (8.1, 7.7, 19.3, 1.6, 635, 345, 0)$.

is obtained only with extreme input values. The appendix contains two theorems which prove that in certain ranges of the parameters this case is not possible. Figure 11 shows the optimal solution value as a function of β, for $(w_1, w_2, \lambda_1, \lambda_2, H_f, \sigma_A, c) = (8.1, 7.7, 19.3, 1.6, 635, 345, 0)$. For these values of the parameters we have $V^{(i)} \leq 0$ for $i = 1, \ldots, 5$. The values of $V_k^{(9)}$ are given in Table 1 for $k = 1, \ldots, 7$.

Future research should address the possible structures of optimal policies for a general number of customers. In particular, an interesting question to be posed is if the optimal policy for a discrete number of customer types greater than 2, is still either simple, semi-continuous, or consisting of a continuous sales interval followed by a number of no-sales interval? If the answer is positive then a further question is the specific policies that may be optimal for each of the possible structures with respect to the customers who are served. The challenging extension assumes a continuous probability distribution $F(w)$ for $w \geq 0$, on the proportion of customers with a reservation price no greater than w. The question then is to develop an algorithm that generates an optimal policy as a function of F.

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Appendix

We have shown that optimal solutions which are neither simple nor semi-continuous are possible only if $\beta > 0.5$ and $p = w_2$. In this appendix we present two theorems restricting the range of input parameters for which it is possible that the optimal solution is neither simple nor semi-continuous.

In this appendix we simplify notation and mark $V_k^{(9)}$ by V_k.

Theorem 15 *If* $\beta \in \left[\frac{1}{2}, \frac{1}{2}\left(1 + \frac{\lambda_2}{\lambda_1}\right)\right]$ *then* $V_1 \geq V_k$ *for* $k \geq 2$.

Proof: Fix T and consider a solution with $k > 1$. The sales at $T-(k-1)\Delta_1$ positively contribute to the profitability of the solution only if $(w_2 - c) \geq h_f[T - (k-1)\Delta_1]$, and therefore we make this assumption.

$$V_k(T) - V_{k-1}(T) = -\frac{\Delta_1}{T}\left\{\lambda_2(w_2 - c) + h_f\lambda\left(-T + \frac{2k-1}{2}\Delta_1\right) + \lambda_1 h_f\left[T - k\Delta_1 + \Delta_1(1-\beta)\right]\right\}.$$

We now apply $w_2 - c \geq h_f[T - (k-1)\Delta_1]$ and use the identity $\lambda = \lambda_1 + \lambda_2$ to obtain

$$V_k(T) - V_{k-1}(T) \leq -\frac{\Delta_1^2 h_f}{2T}[\lambda_2 + \lambda_1(1 - 2\beta)].$$

Thus, if $\beta \leq \frac{1}{2}\left(1 + \frac{\lambda_2}{\lambda_1}\right)$, $V_k(T) - V_{k-1}(T) \leq 0$ and $k = 1$ gives higher profits than any greater value of k. These inequalities hold for any fixed T and therefore also for the respective optimal values. ∎

We are left with the case $\beta > \frac{1}{2}\left(1 + \frac{\lambda_2}{\lambda_1}\right)$, where, in particular, $\lambda_2 < \lambda_1$. The following theorem further limits the possibility that $V_k^{(9)}$ is optimal for some $k \geq 2$.

Theorem 16 *Suppose that* $\lambda_2 < \lambda_1$ *and* $\sigma \leq \frac{1}{2}h_f$ *(equivalently* $\sigma_A \leq \frac{1}{2}H_f$*). For every* $k > 1$*, if* $V_k > 0$ *then* $V_k \leq V^{(3)}$.

Proof: Clearly, $V_k(T)$ decreases with h_f. We also note that it decreases with β: When β grows, there are more sales at T, and the same amount everywhere else, except for that the reduced sale at T is now sold at T_I, so this change increases the holding costs and decreases V. Consequently, we can upper bound $V_k(T)$ by its value at the extreme values $h_f = 2\sigma$ and $\beta = \frac{\lambda}{2\lambda_1}$. For these extremes we get
$E = \lambda(w_2 - c) + (w_1 - w_2)[\lambda_2(2k-1) + \lambda_1]$,
$BC = \lambda\left\{\sigma_A + k(w_1 - w_2)\lambda_2[(w_2 - c) + (k-1)(w_1 - w_2)]\right\}$.

Note that $V^{(3)}$ is independent of β and h_f. We claim that even with these values, $V^{(3)} \geq V_k$, or equivalently

$$\lambda_2(w_1 - w_2)(2k-3) \leq$$

$$2\sqrt{\lambda}\left(\sqrt{\sigma_A + k\lambda_2(w_1-w_2)[(w_2-c)+(k-1)(w_1-w_2)]} - \sqrt{\sigma_A + \lambda_2(w_1-c)(w_1-w_2)}\right),$$

whenever $V_k > 0$ and $\sqrt{C/B} \geq k\Delta_1$. For $k > 1$, the right-hand side of the inequality decreases with σ_A. Therefore, if we prove the inequality for certain value of σ_A then it also holds for smaller values of σ_A. Let

$$\sigma_0 = \frac{(\lambda(w_2-c)+(w_1-w_2)[\lambda_2(2k-1)+\lambda_1])^2}{4\lambda} - k(w_1-w_2)\lambda_2[(w_2-c)+(k-1)(w_1-w_2)].$$

By assumption $V_k > 0$ or equivalently $\sigma_A < \sigma_0$. Hence, it is sufficient to prove that for $\sigma_A = \sigma_0$, $V^{(3)} \geq 0$. Using Figure (4), we need to prove that this σ_A satisfies

$$\text{Diff} = \Big[\lambda(w_1-c)+\lambda_2(w_1-w_2)\Big]^2 - 4\lambda\Big[\sigma_A + \lambda_2(w_1-c)(w_1-w_2)\Big] \geq 0.$$

Equivalently, we prove the inequality after substituting w_i for $w_i - c$, $i = 1, 2$. Substituting the σ_0 for σ_A and simplifying gives

$$\begin{aligned}
\text{Diff} &= \lambda_1\lambda_2\Big\{w_1^2(4k^2-8k+2)+w_2^2(4k^2-8k)-w_1w_2(8k^2+16k-2)\Big\} \\
&\quad +\lambda_2^2\Big\{-w_1^2-3w_2^2+4w_1w_2\Big\} \\
&\geq \lambda_1\lambda_2(w_1-w_2)^2(4k^2-8k)+\lambda_2^2(w_1^2-3w_2^2+2w_1w_2) \geq 0,
\end{aligned}$$

where we used the relations $w_1 \geq w_2$, $\lambda_1 > \lambda_2$, and $k \geq 2$. ∎