A NEW SUFFICIENT CONDITION ON THE TOTAL BALANCEDNESS OF REGULAR CENTRALIZING AGGREGATION GAMES

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Working Paper No 4/2015 June 2015

Research no.: 01220100

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This paper was partially financed by the Henry Crown Institute of Business Research in Israel.

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June 17, 2015

Abstract

We propose a new sufficient condition for total balancedness of regular cooperative games. In a regular game each player is characterized by a "vector of properties" that specifies the initial quantities of a number of resources owned by the player. The characteristic function value of a coalition depends only on the vectors of properties of its members through an algebraic expression. Within this class we focus on aggregation games, where the formation of a coalition is equivalent to aggregating its players into a single "new" player having a cost that is a kind of an average of the costs of the aggregated players. We prove that under some conditions such games are totally balanced and their nonnegative part of the core is fully identifiable. Applications in queueing and scheduling are presented.

Subject classification: Games/Group Decisions: Bargaining, Cooperative; Queues; Production/Scheduling
Area of Review: Games, information, and Networks or Operations and Supply Chain

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1 Introduction

The trigger of this article comes from [1] where we analyzed the total balancedness of the most basic type of cooperation among parallel $M/M/1$ queues that combine their arrival streams as well as their service capacities to form a new $M/M/1$ queue, where the value of a coalition is its congestion. In that paper, the game in question is shown to be non-concave but we propose a concave auxiliary game whose core coincides with the nonnegative part of the core of the original game, proving the total balancedness of the original game. The question that has been often raised in the context of the proof is whether the genuine idea behind the auxiliary game can be generalized beyond the specific game considered in [1]. In this paper we generalize the type of games that the auxiliary game technique is useful in proving total balancedness.

We consider the total balancedness of a class of cooperative games with transferrable utilities, which we call *centralizing aggregation games*. For that sake we first review the main concepts in cooperative game theory. Cooperative games, in general, are coalitional games defined by a pair (N, V) where $N = \{1, \ldots, n\}$ is a set of n *players* and the *characteristic function* V is a set function that returns the cost $V(S) \geq 0$ of any coalition $S \subseteq N$, i.e., $V : 2^N \to \Re_0^+$, ($\Re_0^+$ are the nonnegative real numbers), such that $V(\emptyset) = 0$. The coalition $S = N$ is called the *grand-coalition*. The cost of the game, if the grand-coalition is partitioned into disjoint coalitions $S_1, \ldots, S_m$, so that $\cup_{\ell=1}^m S_\ell = N$ and $S_k \cap S_\ell = \emptyset$ for any $1 \leq k < \ell \leq m$, is $\sum_{\ell=1}^m V(S_\ell)$, meaning that the total cost is additive in the coalitions. A sufficient and necessary condition for all the players of N to fully cooperate in order to form a single coalition that is the grand-coalition, is subadditivity of the game. A game (N, V) is *subadditive* if and only if the characteristic function V is *subadditive*, i.e., for any two disjoint coalitions $S, T \subset N$, $V(S \cup T) \leq V(S) + V(T)$. Subadditivity implies that $V(N) \leq \sum_{\ell=1}^m V(S_\ell)$ for all partitions $\{S_1, \ldots, S_m\}$, $m \geq 1$, of N, implying that the grand-coalition is an optimal formation of coalitions.

Once that the grand-coalition is formed, a bargaining process starts among the players in order to determine how to fairly allocate the cost $V(N)$ among the players. Researchers have proposed various cost allocation concepts, where the common guideline is achieving a reasonable amount of fairness among the players. Let the $n-$ vector $\hat{x} = (x_1, \ldots, x_n) \in \Re^n$, where x_i, $i \in N$, is the cost allocated to player i, be a *cost allocation vector*. The condition $\sum_{i=1}^n x_i = V(N)$, called *efficiency*, is preliminary for an $n-$ vector to be a cost allocation. For the sake of this paper we describe the two most renowned concepts. The *core* of the game (N, V) consists of all efficient cost allocations that allocate to the members of S, for any coalition $S \subset N$, no more than $V(S)$, i.e., $\sum_{i \in S} x_i \leq V(S)$. These last conditions are called the *stand-alone* conditions. The core of a game (N, V) is thus defined by n decision variables and $2^n - 1$ linear constraints. As a consequence, the core is either empty, consists of a single cost allocation or it contains an infinitely large set of cost allocations. A cooperative game (N, V) whose core is nonempty is said to be *balanced*, and a game whose core and the cores of all its sub-games are nonempty is called *totally balanced.* Finding out whether a given game is balanced or totally balanced or alternatively pointing out a specific cost allocation within the core might be challenging questions because of the exponential size of the problem. Another well known fairness concept is the *Shapley value*. Any cooperative game is associated with a unique value, called its Shapley value, even if the core of the game is empty. The Shapley value of player $i \in N$ is the average marginal cost of adding the player to the players that precede him where averages are taken with respect to all potential orders of the players, see [8]. The Shapley value does not necessarily belong to the core of a balanced game as it does not necessarily satisfy all the stand-alone conditions. On the other hand, the Shapley value does satisfy three other desired properties of symmetry, linearity and null player, in addition to the efficiency, see [8]. In fact, the Shapley value is the only value that satisfies all the above four properties.

Many cooperative games, especially in operations management and logistics, have a further feature that makes it possible to represent them in a more efficient and compact way than by listing the 2^n values that the characteristic function assumes. In two former papers we propose a class of cooperative games called *regular games* see [2] and [3], which are defined by a finite list of $\kappa \geq 1$ different quantitative resources indexed by $\ell = 1, \ldots, \kappa$ that the players own. Each player is fully characterized by a *vector of properties* of size κ whose $\ell-$th element represents the amount of resource ℓ that the player owns. The cost of a coalition is a symmetric mapping of the vectors of properties of the players in the coalition into $\Re_0^+$ and it is otherwise independent of n or the identity of the players. Regular games can be presented in a simple and compact way by stating the form of the mapping as a function of the collection of vectors of properties. As a consequence it allows flexibility that the classic presentation (N, V) does not permit, as instead of having a rigid set of players N, the mapping returns a value for any collection of $\kappa-$vectors even if the vectors are not associated with players in N. Furthermore, manipulating real functions using appropriate mathematical rules in order to prove desired properties is simpler and a safer haven than doing the same with set functions. For example, consider the following definition, see [2], on homogenous of degree p, $p \geq 0$, cooperative games:

Definition 1 *A regular game is homogeneous of degree* p, $p \geq 0$, *if for any integer* m, *the cost of cloning* m *times a collection of vectors of properties, is* m^p *times the cost of the original collection of vectors of properties.*

In a non-regular game (N, V), cloning a player is meaningless as the characteristic function is defined only on coalitions of N. A regular game that is homogenous of degree 0 displays *economies of scale* as the cost of $m \geq 1$ copies of a collection of vectors of properties is the same as the cost of a single copy, thus the cost per player decreases implying that efficiencies are improved by scale. In homogenous of degree 1 games the cost increases linearly in the number of copies of the players, thus there are not any economies of scale. Homogenous of degree p games for $0 < p < 1$ ($p > 1$) display economies (diseconomies) of scale, as the average cost per player improves (deteriorates). Paper [2] exploits the advantages of regular games in proving a new sufficient condition for a game to be totally balanced, see Condition 3 below. The regularity of the game allows to prove the condition by applying arithmetic operations on the cost function rather than dealing with the characteristic function.

Below we list the few known sufficient conditions for proving total balancedness of a game. The first two conditions hold in general, where the third is applicable only on regular games.

- **Condition 1.** A game $G = (N, V)$ is *concave* if its characteristic function is concave, meaning that for any two coalitions $S, T \subseteq N$, $V(S \cup T) + V(S \cap T) \leq V(S) + V(T)$. Concave games are subadditive but not the other way around. It was shown in [9] that the core of a concave game possesses $n!$ extreme points, each of which being the vector of marginal contribution of the players to a different permutation of the players. In particualr, the Shapley value of a concave game is the center of gravity of its core.

  Remark 1. The authors of [4] introduce the set of *average concave games* that contains as a subset the concave games. These games are totally balanced and their Shapley value is within their core, similarly to concave games.

- **Condition 2.** A *market game*, see e.g., Chapter 13 in [5], is defined as follows: Suppose there are κ inputs. An *input vector* is a nonnegative vector in $(\Re_0^+)^\kappa$. Each of the n players possesses an initial commitment vector $w_i \in (\Re_0^+)^\kappa$, $1 \leq i \leq n$, which states a nonnegative quantity for each input. Moreover, each player is associated with a continuous and convex cost function $f_i : (\Re_0^+)^\kappa \rightarrow \Re_0^+$, $1 \leq i \leq n$. A profile $(z_i)_{i \in N}$ of input vectors for which $\sum_{i \in N} z_i = \sum_{i \in N} w_i$ is an *allocation.* The game is such that a coalition S of players seeks an optimal redistribution of its members' commitments among its members in order to get a profile $(z_i)_{i \in S}$ of input vectors that minimizes the total cost across the members of S. Formally, for any $\emptyset \subseteq S \subseteq N$,

  $$V(S) = \min \Big\{ \sum_{i \in S} f_i(z_i) : z_i \in (\Re_0^+)^\kappa, i \in S \text{ and } \sum_{i \in S} z_i = \sum_{i \in S} w_i \Big\} \qquad (1)$$

  Market games are known to be totally balanced, see [6], Corollary 3.2.4. However, in contrast to concave games whose entire core is well defined, just a single core cost allocation that is based on *competitive equilibrium prices* is known for market games, see [5] page 266.

  It is interesting to note that in [10] it is proved that a game is totally balanced if and only if it is a market game. In particular, it means that all concave games are market games. In addition it means that any totally balanced game has a presentation as a market game (see (1)), even if originally it is not presented in this way. The drawback is that the attempt to prove total balancedness by representing a game by a characteristic function that satisfies (1) may be as challenging as the task of proving that the game is totally balanced.

- **Condition 3.** A regular game $G = (N, V)$ that is subadditive and homogenous of degree 1 is totally balanced, see [2].

  No core allocation is proposed in [2] for a subadditive and homogenous of degree 1 game. However, the fact that it is totally balanced implies that a core cost allocation based on competitive equilibrium prices can be derived once that the game is presented as a market game.

We note that neither concave games nor market games are necessarily regular. A market game (N, V) is a regular game if and only if the characteristic function value of the grand-coalition $V(N)$, (see (1)), is given as the minimum of the sum of n identical functions, i.e., $f_i = f$ for all $i \in N$. In such a case, the cost of a vector of properties is independent of the identity of the player that owns the vector.

In this article we concentrate on a subclass of regular games that we call *centralizing aggregation games*, where an *aggregation function* aggregates any number of vectors of properties into a single new vector of properties. Centralizing means that the cost of the aggregated vector of properties generated by an input of vectors of properties behaves like a measure of centrality of the costs of the individual vectors of the input. More specifically, the cost of the new vector is in between the cost of the cheapest vector and the cost of the most expensive vector in the input, and it is strictly increasing in the cost of the vectors in the input. Note that the aggregated vector is not necessarily associated with a player of N, but nevertheless, as the game is regular, the cost of the aggregated vector is well defined.

In the main theorem of the paper we prove that under a certain condition, a centralizing aggregation game is totally balanced and its nonnegative core is fully identifiable. This is done by defining an auxiliary game whose core is contained in the core of the original game, and showing that the auxiliary game is monotone and concave thus its core is nonnegative, see [9]. Finally, we show that the core of the auxiliary game coincides with the nonnegative core of the original game.

The outline of the paper is as follows: In Section 2 we present some notations and preliminaries, including a rigorous definition of regular games. In Section 3 we present the class of regular centralizing aggregation games and some of their properties that are required for proving the main theorem in Section 4. These results are used in Section 5 to show that two specific games, one in queueing and one in scheduling, are totally balanced. Section 6 summarizes the paper and raises an open question about the total balancedness of a family of regular centralizing aggregation games whose characteristic function returns, for any coalition, a generalized mean of the scores of its members.

2 Notations and Preliminaries

In this article we focus on a class of cooperative games, called *regular games*, see [2], and [3], described hereafter. In a regular game $G = (N, V)$, with a set of n players $N = \{1, \ldots, n\}$, the players own an integer number $\kappa \geq 1$ of resources indexed by $\ell = 1 \ldots, \kappa$. Each player $i \in N$ is associated with a vector $y^i \in D \subset \Re^\kappa$, called its *vector of properties* where y^i_ℓ specifies the amount of resource ℓ, $1 \leq \ell \leq \kappa$, that is initially owned by the player. The set D is assumed to be a convex set. The set D may coincide with $\Re^\kappa$ if any real $\kappa-$vector is a possible vector of properties, though usually D is a subset of $(\Re^+_0)^\kappa$. The resources can be reallocated among the players of a coalition according to the rules of the game. In a regular game the cost induced by any coalition $S \subseteq N$ consisting of $s \geq 1$ players, namely the characteristic function value $V(S)$, is independent of n or the players' identity, and it depends only on the s vectors of properties that are associated with the players of S. More specifically, $V(S)$ can be represented as a symmetric mathematical expression of the s vectors of properties of the members of S, i.e., $V_s : D^s \rightarrow \Re^+_0$. In particular, for the empty coalition, $V_0 \equiv 0$, and for singleton coalitions $V(\{i\}) = V_1(y^i)$ for any $i \in N$. A *null vector of properties* denoted by $y^0 \in D$, whose cost is zero, i.e., $V_1(y^0) = 0$, exists. However, these requirements are insufficient for a game to be a regular game, as we further need the various functions in the infinite sequence $V_0, V_1 \ldots$ to be interrelated. This interrelation is achieved by using the null vector of properties, as explained below.

Let $y^{(s)}$ denote a sequence of s vectors of properties $y^1, \ldots, y^s$ in D. The following two definitions formally define a regular game:

Definition 2 *An infinite sequence of symmetric functions $V_0, V_1, \ldots, V_m, \ldots$ is said to be Infinite Increasing Input-Size Symmetric Sequence (IIISSS) of functions for given integer $\kappa \geq 1$, and a convex set D of $\Re^\kappa$, if*

- $V_0 \equiv 0$;
- *For any $m \geq 1$, $V_m : D^m \rightarrow \Re^+_0$;*
- *There exists a vector $y^0 \in D$ such that $V_1(y^0) = 0$ and for any given sequence of $m-1$ vectors of properties $y^{(m-1)} = (y^1, \ldots, y^{m-1}) \in D^{m-1}$, $V_{m-1}(y^{(m-1)}) = V_m(y^{(m-1)}, y^0)$.*

The third item of the definition guarantees that the IIISSS of functions is consistent, i.e., it excludes the possibility that there exist two functions V_ℓ and V_k for $\ell \neq k$, where each is defined by a different mathematical expression. The *null vector of properties* $y^0 \in D$ links the different functions through a forward recursion.

Definition 3 *A game $G = (N, V)$ is called regular if there exists a convex set $D \subseteq \Re^\kappa$, such that each player i, $i \in N$, is associated with a vector of properties $y^i \in D$, and there exists an IIISSS of functions $V_\ell : D^\ell \rightarrow \Re^+_0$, $\ell \geq 0$, such that for any $S \subseteq N$, $V(S) = V_{|S|}(y^i|_{i\in S})$.*

A regular game is easily extendable to any set of players once that each player is associated with a vector of properties. As a consequence, it is possible to duplicate players as shown in Definition 1 and in the proof of the main theorem in paper [2]. The class of regular games is quite large and it contains many well-known games in economics, operations and service management, graph theory etc. As stated in Section 1, a market game (N, V) is a regular game if and only if $V(N)$, see (1), is given as the minimum of the sum of n identical functions, i.e., $f_i = f$ for all $i \in N$. Consider a market game where the cost of player i that owns a vector of properties $y^i \geq 0$ is $f(y^i)$. In order to define it as a regular game, let D be $(\Re_0^+)^\kappa$, y^0 is the zero-vector in $\Re^\kappa$, and $V_m(y^1, y^2, \ldots, y^m) = \min\{\sum_{i=1}^m f(z^i) : \sum_{i=1}^m z^i = \sum_{i=1}^m y^i\} = mf(\frac{\sum_{i=1}^m y^i}{m})$. The last equality follows from the convexity of f, see definition of a market game in Section 1.

The next definition is general:

Definition 4 *A cooperative game (N, V) is monotone non-decreasing if and only if $V(T) \geq V(S)$ for any $S \subset T \subseteq N$.*

In the sequel we say that a game is monotone if it is monotone non-decreasing.

3 Centralizing Aggregation Games

We now present a few more definitions.

Definition 5 *The IIISSS of functions $\{V_k\}_{k \geq 0}$ of a regular game (N, V) is centralizing if for any $m \geq 1$ vectors of properties $y^1, \ldots, y^m$ with $V_1(y^1) \leq V_1(y^2) \leq \ldots \leq V_1(y^m)$, the following two properties hold:*

- *$V_1(y^1) \leq V_m(y^1, \ldots, y^m) \leq V_1(y^m)$, and*
- *$V_m(y^1, \ldots, y^{m-1}, z)$ is strictly increasing in $V_1(z)$, for $z \in D$.*

We now proceed to the definition of aggregation games:

Definition 6 *An aggregation function maps a $2-$fold cartesian product of a convex set into the same set, given that it is reflexive, symmetric and it satisfies the commutative and the associative laws.*

Thus, an aggregation function g, $g : D^2 \to D$, satisfies the following equalities: for any $y^i, y^j, y^k \in D$, $g(y^i, y^i) = y^i$, $g(y^i, y^j) = g(y^j, y^i)$, and $g(g(y^i, y^j), y^k) = g(y^i, g(y^j, y^k))$. Let $g^{m-1} : D^m \to D$ be the aggregation function that aggregates $m \geq 2$ vectors of properties into one, where $g^1 \equiv g$. In other words, g^{m-1} is the $m-$fold aggregation function.

Next, we consider an IIISSS of functions and an aggregation function $g : D^2 \to D$ that satisfy $V_m(y^1, \ldots, y^m) = V_1(g^{m-1}(y^1, \ldots, y^m))$.

Definition 7 *A game (N, V) is an aggregation game if*

- *It is a regular game with an IIISSS of functions $\{V_k\}_{k \geq 0}$.*
- *There exists an aggregation function $g : D^2 \to D$ such that the IIISSS of functions satisfies $V_m(y^1, \ldots, y^m) = V_1(g^{m-1}(y^1, \ldots, y^m))$ for any $m \geq 1$ vectors of properties $y^1, \ldots, y^m \in D$.*

Thus, a regular aggregation game (N, V) associated with an IIISSS of functions $\{V_k\}_{k\geq 0}$, is fully characterized by its aggregation function $g : D^2 \to D$ and the cost function $V_1 : D \to \Re_0^+$.

Definition 8 *An aggregation game (N, V) that is associated with a centralizing IIISSS of functions $\{V_k\}_{k\geq 0}$ is a centralizing aggregation game.*

The next claim provides an alternative definition to a centralizing aggregation game that is given in terms of V_1 and the aggregation function g. The proof of the claim is by induction using the properties of the aggregation function.

Claim 1 *An aggregation game defined by the aggregation function $g : D^2 \to D$ and the cost function $V_1 : D \to \Re_0^+$ is a centralizing aggregation game if and only if*

- *$V_1(y) \leq V_1(z)$ implies that $V_1(y) \leq V_1(g(y, z)) \leq V_1(z)$.*
- *$V_1(g(y, z))$ is strictly increasing in $V_1(z)$ or any $y, z \in D$.*

For demonstration of centralizing aggregation games, we present two examples where the characteristic function is either the arithmetic mean or the geometric mean of the scores. The first example is presented also in [2]. The games are easily verified to be centralizing aggregation games by Claim 1. Let $\mathcal{N} = \{1, 2, \ldots\}$ be the set of positive integers, and $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$.

- The arithmetic mean game: Each player i is associated with a nonnegative score α_i and the value of a coalition is the average score of its members. In order to define the game as a regular aggregation game let $\kappa = 2$, $y^0 = (0, 0)$, $D = \{(0, 0)\} \cup \{(x, \ell) : x \geq 0, \ell \in \mathcal{N}\}$ and each individual player is associated with a vector of properties $(a, 1)$ where a is the score of the player. In addition, let the aggregation function $g : D^2 \to D$ be $g((x_1, k_1), (x_2, k_2)) = (x_1 + x_2, k_1 + k_2)$. Each group of k original players is associated with a vector of properties of the form (x, k) where x is the sum of the scores of the group's players. Let $V_1(y^0) = 0$ and otherwise, for any $(x, k) \in D$, $(x, k) \neq y^0$ let $V_1((x, k)) = \frac{x}{k}$. Given a collection of $m \geq 1$ vectors of properties $y^{(m)}$ in D^m, with $y^i = (x_i, k_i) \in y^{(m)}$, $i = 1, \ldots, m$, let $V_m(y^{(m)}) = V_1(g^{m-1}(y^{(m)})) = \sum_{i=1}^m x_i / \sum_{i=1}^m k_i$ if $\sum_{i=1}^m k_i > 0$, i.e., $V_m(y^{(m)})$ is the average score of the non-null vectors in D, and otherwise, it is zero. In subsection 5.2 we show that this game is totally balanced.
- The geometric mean game: Each player i is associated with a positive score α_i and the value of a coalition is the geometric mean of the scores of its members. For example, the value of a coalition of three players is the cube root of the product of their three scores. Thus, let $\kappa = 2$ and similarly to the average game, each individual player is associated with a vector of properties $(a, 1)$ where a is the score of the player. However, unlike the average game, the null vector of properties here is $y^0 = (1, 0)$ and $D = \{(1, 0)\} \cup \{(x, \ell) : x > 0, \ell \in \mathcal{N}\}$. Let the aggregation function $g((x_1, k_1), (x_2, k_2)) = (x_1 x_2, k_1 + k_2)$. Each

group of k original players is associated with a vector of properties of the form (x,k) where x is the product of the scores of the group's players. Let $V_1(y^0) = 0$ and otherwise, for any $(x,k) \in D$, $(x,k) \neq y^0$, let $V_1((x,k)) = x^{1/k}$. Given a collection of $m \geq 1$ vectors of properties $y^{(m)}$ in D^m, $y^i = (x_i, k_i) \in y^{(m)}$, $i = 1, \ldots, m$, the corresponding IIISSS of functions is given by $V_1(y^0) = 0$, and otherwise $V_m(y^{(m)}) = V_1(g^{m-1}(y^{(m)})) = (\Pi_{i=1}^m x_i)^{(\sum_{i=1}^m k_i)^{-1}}$, i.e., $V_m(y^{(m)})$ is the geometric mean score of the non-null vectors of properties in D.

As shown above, the choice of the null vector of properties y^0 depends on the IIISSS of functions that should satisfy the third requirement in Definition 2. In Section 6 we pose an open question regarding the total balancedness of a large family of centralizing aggregation games that deal with *generalized means* that the arithmetic and geometric mean games are special cases. The maximum of a set of numbers is another a special case of a generalized mean. Consider a game where each player is associated with a score and the characteristic function value of any coalition is the maximum score in the coalition. This game, which we call the *maximum game*, can easily be presented as a regular aggregation game, but it is not a centralizing game as it does not satisfy the second item of Claim 1. I.e., $V_1(g(y,z))$ for $y, z \in D$, is increasing but not strictly increasing in $V_1(z)$, as required. For more on the maximum game see Claim 3.

Claim 2 *A centralizing aggregation game* (N,V) *is subadditive, homogenous of degree 0 and nonmonotone.*

Proof:

In order to prove the subadditivity, consider two disjoint coalitions $S, T \subset N$, with $V(S) \leq V(T)$. The centralizing property implies that $V(S) \leq V(S \cup T) \leq V(T)$, which implies that $V(S \cup T) \leq V(S) + V(T)$. Such a game is homogenous of degree 0, see Definition 1, as cloning the players of any coalition $S \subseteq N$ m times results in m coalitions $S^1, \ldots, S^m$ where each such coalition is a copy of S, thus $V(S^1) = \ldots = V(S^m) = V(S)$. The centralizing property implies that $V(S) = V(S^1) \leq V(S^1 \cup \ldots \cup S^m) \leq V(S^m) = V(S)$, i.e., $V(S^1 \cup \ldots \cup S^m) = V(S)$. Finally, regarding the nonmonotonicity, consider any two vectors of properties $y^1, y^2 \in D$ such that $V_1(y^1) < V_1(y^2)$. By Claim 1, $V_1(y^1) = V_1(g(y^1, y^1)) < V_1(g(y^1, y^2)) < V_1(g(y^2, y^2)) = V_1(y^2)$, proving that adding a player to a coalition may increase or decrease the cost of the coalition.

■

Consider now the class of nonmonotone games, not necessarily regular, that contains as a subset the regular centralizing aggregation games.

Definition 9 *Any nonmonotone game* $G = (N,V)$ *is associated with another monotone game* $\tilde{G} = (N, \tilde{V})$ *called its auxiliary game where* $\tilde{V}(S) = \min\{V(T) : S \subseteq T \subseteq N\}$.

Note that the auxiliary game of a monotone game coincides with the game itself. The following Lemma follows from [1].

Lemma 1 • *The auxiliary game $\tilde{G} = (N, \tilde{V})$ of a nonmonotone game $G = (N, V)$ is a monotone game, with $\tilde{V}(\emptyset) = 0$, $\tilde{V}(N) = V(N)$, and $\tilde{V}(S) \leq V(S)$.*

- *If the auxiliary game $\tilde{G} = (N, \tilde{V})$ is totally balanced, then the game $G = (N, V)$ is also totally balanced.*
- *If the auxiliary game $\tilde{G} = (N, \tilde{V})$ is concave, then the nonnegative core of the game $G = (N, V)$ coincides with the core of the auxiliary game.*

Definition 10 *Coalition $T \subset N$ is minimal for coalition $S \subseteq T$ for a given game (N, V) if and only if $\tilde{V}(S) = V(T)$.*

A minimal coalition is not necessarily unique. In [1] it is proved that the maximal coalition among the minimal ones is unique, and a Construction Algorithm that generates the maximal coalition is presented. In the sequel let $\tilde{S}$ be the maximal coalition among all minimal coalitions of $S \subseteq N$.

An alternative definition for a concave game, see Section 1, Condition 1, is given in Definition 11. In the sequel let $S \cup \{\ell\} = S_\ell$ for any coalition $S \cup \{\ell\} \subset N$.

Definition 11 *A cooperative game (N, V) is concave if and only if it satisfies the following property for any $S \subset T \subset N$ and for any $\ell \in N \backslash T$:*

$$V(T_\ell) - V(T) \leq V(S_\ell) - V(S).$$

The next claim states that the maximum game is concave.

Claim 3 *Consider a set of players N where each player $i \in N$ is associated with a score $\alpha_i \in \Re$. The maximum game (N, V) where $V(S) = \max_S\{\alpha_i\}$ is monotone and concave. Moreover, the core of the maximum game is nonnegative.*

Proof: The maximum game is monotone as adding a player to a coalition can cannot reduce the cost. The proof that the maximum game is concave follows directly from Definition 11. As specified in Condition 1, see Section 1, and [9], in concave games each cost allocation vector that is an extreme point of the core is associated with a certain permutation of the players, so that each player is allocated a cost that is equal to the marginal cost of adding the player to all the players that precede him according to the permutation. Thus, the core of a monotone concave game is nonnegative. ∎

Concave games are the most structured cooperative games whose core is fully characterized, see [9] and Condition 1 in Section 1. According to Lemma 1, if the auxiliary game of a nonmonotone game is concave, then the original game is totally balanced and the core of the auxiliary game coincides with the nonnegative part of the core of the original game. This result is in contrast to the result of [2] that provides just a sufficient condition for a game to be totally balanced without specifying any cost allocation in the core. Even for market games that are known to be totally balanced, just a single core allocation based on competitive equilibrium prices is known. We will show that under the sufficient conditions presented in Theorem 1 in Section 4, the whole nonnegative core of a centralizing aggregation game is fully characterized. The sufficient conditions are based on the function $V_1 \circ g : D^2 \to \Re_0^+$. For that sake we need the next definition:

Definition 12 *The composition $V_1 \circ g : D^2 \to \Re_0^+$ of an aggregation game $G = (N, V)$ is said to have decreasing differences if for any three vectors of properties $x, y, z \in D$ such that $V_1(x) < V_1(y) \le V_1(z)$, the following inequality holds: $V_1(g(y, z)) - V_1(y) \le V_1(g(x, z)) - V_1(x)$.*

In centralizing aggregation games both sides of the last inequality are nonnegative. Before concluding this section, we consider a few basic observations. The first two consider ratios of nonnegative numbers, where the proof of the first one is direct. We will use these observations in Section 5 where we present the examples.

Observation 1 *Let $a_i, a_j \ge 0$ and $b_i, b_j > 0$. $\frac{a_i}{b_i} \le \frac{a_j}{b_j}$ if and only if $\frac{a_i}{b_i} \le \frac{a_i+a_j}{b_i+b_j} \le \frac{a_j}{b_j}$. Moreover, $\frac{a_i}{b_i} < \frac{a_j}{b_j}$, if and only if $\frac{a_i}{b_i} < \frac{a_i+a_j}{b_i+b_j} < \frac{a_j}{b_j}$.*

Observation 2 *If $a_i, a_j, a_k \ge 0$ and $b_i, b_j, b_k > 0$ such that $\frac{a_i}{b_i} < \frac{a_j}{b_j} \le \frac{a_k}{b_k}$, then $0 \le \frac{a_j+a_k}{b_j+b_k} - \frac{a_j}{b_j} \le \frac{a_i+a_k}{b_i+b_k} - \frac{a_i}{b_i}$.*

Proof:

Consider the function $\phi(\alpha, \beta) = \frac{\alpha}{\beta}$ for $\alpha \ge 0$ and $\beta > 0$. Proving the observation is equivalent to proving that the function $\phi(\alpha, \beta)$ has decreasing differences. For that sake consider the function $\psi(\alpha, \beta) = \frac{\alpha+\theta}{\beta+\delta} - \frac{\alpha}{\beta}$, where $\theta \ge 0$, $\delta > 0$ are fixed constants, and $\frac{\alpha}{\beta} < \frac{\theta}{\delta}$. Rewrite the function $\psi(\alpha, \beta)$ as a function π of the two variables β and $\rho = \frac{\alpha}{\beta}$: $\pi(\beta, \rho) = \frac{\rho+\theta\beta^{-1}}{1+\delta\beta^{-1}} - \rho$. It remains to verify that the function $\pi(\beta, \rho)$ is decreasing in ρ. This is done by checking the sign of the first partial derivative of π with respect to ρ,

$$\frac{\partial \pi(\beta, \rho)}{\partial \rho} = \frac{1}{1 + \delta\beta^{-1}} - 1 < 0, \tag{2}$$

completing the proof.

■

Note that Observations 1,2 hold for the arithmetic mean game by substituting the second entry in the vector of properties of each individual player $i \in N$ by 1.

Observation 3 *Consider a game (N, V), where $V(S) = \max\{U^1(S), U^2(S), \ldots, U^L(S)\}$, $L \ge 2$, for any coalition $S \subseteq N$. If each of the games (N, U^k) for $k = 1 \ldots L$, is totally balanced, then, the game (N, V) is also totally balanced. Moreover, let $k^* = \arg\max\{U^k(N) : k = 1 \ldots L\}$, then the core of the game (N, U^{k^*}) is a subset of the core of the game (N, V).*

Proof: Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be a core cost allocation of the game (N, U^{k^*}). We will show that $\vec{\alpha}$ is a core cost allocation also for the game (N, V). The efficiency property of the cost allocation $\vec{\alpha}$ holds as $\sum_{i=1}^{n} \alpha_i = V(N)$ follows from the choice of k^*. For any proper coalition $S \subset N$, the stand-alone property for (N, V) holds as $\sum_{i \in S} \alpha_i \leq U^{k^*}(S) \leq \max\{U^k(S) : k = 1 \ldots L\} = V(S)$.

∎

We now proceed to the main theorem of the paper.

4 Main Theorem

Theorem 1 *Consider a regular nonmonotone centralizing aggregation game (N, V) defined by V_1 and the aggregation function $g : D^2 \to D$ and let its auxiliary game be $\tilde{G} = (N, \tilde{V})$. If the composition $V_1 \circ g : D^2 \to \Re_0^+$ has the property of decreasing differences (see Definition 12) then:*

- *The auxiliary game $\tilde{G} = (N, \tilde{V})$ is monotone and concave, and therefore its core is nonempty and nonnegative.*
- *The game (N, V) is totally balanced.*
- *The core of the auxiliary game $(N, \tilde{V})$ coincides with the nonnegative part of the core of the game (N, V).*

Proof: Monotonicity follows from the definition of the auxiliary game. By Condition 1 in Section 1 and Lemma 1 it is sufficient to show that the auxiliary game $\tilde{G} = (N, \tilde{V})$ associated with this game is concave. We prove the concavity of $\tilde{V}$ by using Definition 11 and showing that

$$\tilde{V}(T_\ell) - \tilde{V}(T) \leq \tilde{V}(S_\ell) - \tilde{V}(S) \quad \text{for any } S \subset T \subset T_\ell \subset N.$$

First note that by the first item of Lemma 1 the auxiliary game $(N, \tilde{V})$ is monotone, implying that both sides of the above inequality are nonnegative. Moreover, if $\ell \in \tilde{T}$ then $\tilde{V}(T_\ell) - \tilde{V}(T) = 0$ and the proof is trivial. Thus we assume that $\ell \notin \tilde{T}$. For this case, we prove a tighter inequality than the above one in three steps:

1. We show that the l.h.s. of the inequality satisfies
$$\tilde{V}(T_\ell) - \tilde{V}(T) \leq V(\tilde{T} \cup \{\ell\}) - V(\tilde{T}).$$
2. We show that the r.h.s. of the inequality satisfies
$$\tilde{V}(S_\ell) - \tilde{V}(S) \geq V(\widetilde{S}_\ell) - V(\widetilde{S}_\ell \backslash \{\ell\}).$$
3. We conclude the proof by showing that
$$V(\tilde{T} \cup \{\ell\}) - V(\tilde{T}) \leq V(\widetilde{S}_\ell) - V(\widetilde{S}_\ell \backslash \{\ell\}).$$

For the first item note that $\tilde{V}(T_\ell) = V(\widetilde{T_\ell}) \leq V(\tilde{T} \cup \{\ell\})$ where the equality follows from the definition of $\widetilde{T_\ell}$ and the inequality follows from the fact the coalition $\tilde{T} \cup \{\ell\}$ contains T_ℓ but it is not necessarily one of its minimal coalitions (see Definition 10), where $\widetilde{T_\ell}$ is. Also, $\tilde{V}(T) = V(\tilde{T})$ by definition.

In order to prove the second item note that $\tilde{V}(S_\ell) = V(\widetilde{S_\ell})$ by definition, and $\tilde{V}(S) \leq V(\widetilde{S_\ell} \backslash \{\ell\})$ as coalition $\widetilde{S_\ell} \backslash \{\ell\}$ contains coalition S but is not necessarily a minimal coalition of S.

Finally, in order to show that $V(\tilde{T} \cup \{\ell\}) - V(\tilde{T}) \leq V(\widetilde{S_\ell}) - V(\widetilde{S_\ell} \backslash \{\ell\})$, let (i) u^1 be the vector of properties obtained by aggregating all the vectors of properties of $\widetilde{S_\ell} \backslash \{\ell\}$ by using repeatedly the aggregation function g, (ii) u^2 be the vector of properties obtained by aggregating all the vectors of properties of $\tilde{T}$ by using repeatedly the aggregation function g, and (iii) $z = y^\ell$ be the vector of properties of player ℓ. Recall our assumption that $\ell \notin \tilde{T}$. As the game is a centralizing game it means that $V(\tilde{T}) < V(\{\ell\})$, or equivalently $V_1(u^2) < V_1(z)$. Note that $S_\ell \subset \tilde{T} \cup \{\ell\}$ and therefore, $V(\widetilde{S_\ell}) \leq V(\tilde{T} \cup \{\ell\})$, where $V(\widetilde{S_\ell}) = V_1(g(u^1, z))$ and $V(\tilde{T} \cup \{\ell\}) = V_1(g(u^2, z))$. Thus, $V_1(g(u^1, z)) \leq V_1(g(u^2, z))$. This last inequality together with the fact that the game is centralizing imply that $V_1(u^1) \leq V_1(u^2)$. It remains to show that $V_1(g(u^2, z) - V_1(u^2) \leq V_1(g(u^1, z)) - V_1(u^1)$, which follows directly from the decreasing differences property of the composition $V_1 \circ g : D^2 \to \Re_0^+$. ■

Note that under the conditions of Theorem 1 the game (N, V) is not necessarily concave, as is the case with the game in [1] that is presented in Sub-section 5.1.

5 Examples

In this section we present two examples of cooperative games. The first is the queueing model that triggered this research, and the second is in scheduling.

5.1 Cooperation of $M/M/1$ queueing system

In the model presented in [1], servers can cooperate in order to minimize the total congestion. When a set of servers cooperate, they form a single server whose service rate is the sum of the individual service rates, and its stream of arrivals is the union of the respective streams of arrivals. More precisely, let $N = \{1, \ldots, n\}$ be a set of n $M/M/1$ queueing systems. Queueing system i is associated with an exponential service rate μ_i and a Poisson arrival rate λ_i, $\lambda_i < \mu_i$, $i \in N$. Cooperation of a set $S \subseteq N$ results in a single $M/M/1$ queue whose capacity is $\mu(S) = \sum_{i \in S} \mu_i$, and its arrival rate is $\lambda(S) = \sum_{i \in S} \lambda_i$. The congestion of any coalition $S \subseteq N$ is given by

$$V(S) = \frac{\lambda(S)}{\mu(S) - \lambda(S)}. \tag{3}$$

Next we present this game as a regular aggregation game: each queueing system is associated with a vector of properties of size $\kappa = 2$, $y^0 = (0, 0)$, and $D = \{0, 0\} \cup \{(\lambda, \mu) | 0 \leq \lambda < \mu\} \subset (\Re^+)^2$. Let $V_1(y^0) = 0$, and for $(\lambda, \mu) \in D \backslash \{0, 0)\}$,

$V_1((\lambda,\mu)) = \frac{\lambda}{\mu-\lambda}$. The aggregation function g that combines two vectors of properties in D into one is $g((\lambda_1,\mu_1),(\lambda_2,\mu_2)) = (\lambda_1+\lambda_2,\mu_1+\mu_2)$. Consider m vectors of properties $y^1,\ldots,y^m$ where $y^i = (\lambda_i,\mu_i)$ for $i = 1\ldots m$. Thus $g^{m-1}(y^1,\ldots,y^m) = (\sum_{i=1}^m \lambda_i, \sum_{i=1}^m \mu_i)$ and $V_m(y^1,\ldots,y^m) = V_1(g^{m-1}(y^1,\ldots,y^m))$. If $g^{m-1}(y^1,\ldots,y^m) \neq y^0$, then $V_m(y^1,\ldots,y^m) = \sum_{i=1}^m \lambda_i / \sum_{i=1}^m(\mu_i-\lambda_i)$, and otherwise it is zero.

It was shown in [1] that this game is nonmonotone and non-concave. However, its auxiliary game was proved to be concave and therefore it is totally balanced. By the third item of Lemma 1, the original game is totally balanced and the non-negative part of its core coincides with the core of the auxiliary game. The proof in [1] is based on showing that the set function defined by Equation (3) is concave. Alternatively, as explained below, Theorem 1 allows a much simpler proof based on real functions only.

Note that each of the IIISSS of functions $\{V_k\}_{k\geq 1}$ is a ratio of a non- negative real number by a positive real number. Thus, by Observation 1, the IIISSS of functions is centralizing. By Theorem 1 we further need to show that the composition $V_1 \circ g : D^2 \to \Re_0^+$ has the decreasing differences property, which follows directly from Observation 2.

5.2 Minimizing Makespan with Preemptions

Consider a number of production units $i \in N = \{1,\ldots,n\}$, hereafter players, where each player i owns $k_i \geq 1$ machines, and is associated with a set of jobs where its longest job is of duration $q_i \geq 0$ and the total processing time of all its jobs is $p_i \geq q_i$. The machines of all players are assumed to be parallel, identical and they have the same speed. Each player schedules its jobs on its machines so that its makespan is minimized. The schedules allow for preemptions, but a job cannot be processed simultaneously on different machines. The optimal solution to the scheduling problem of each player is known, see [7] Chapter 5: The minimum makespan of a player is the maximum of its longest job and the ratio between its total processing time and its number of machines. If the makespan is determined by the longest processing time, then it is possible that the optimal schedule on some of the machines include idle times. Otherwise, all machines are fully used during the makespan duration. In the cooperative game proposed here, players can form coalitions, where the players of a coalition share their machines in order to produce their jobs in a minimum makespan. For any coalition $S \subseteq N$, let $k(S) = \sum_{i\in S} k_i$ be the number of machines owned by the players of S, $q(S) = \max\{q_i : i \in S\}$, be the duration of the longest job in S, and $p(S) = \sum_{i\in S} p_i$, the total processing time of all jobs in S. Clearly, $p(S) \geq q(S)$. If $q(S) = 0$ then also $p(S) = 0$ as it means that coalition S has no jobs. Let (N,V) be the respective game where the characteristic function value $V(S)$ for $S \subseteq N$ denotes the optimal makespan of running the jobs of S on its machines. More specifically, $V(S) = \max\{q(S), \frac{p(S)}{k(S)}\}$. If the players break into m disjoint coalitions $S_1,\ldots,S_m$, such that $N = \cup_{\ell=1}^m S_\ell$, then the total cost is $\sum_{\ell=1}^m V(S_\ell)$. This game is clearly subadditive.

Next we represent the game as a regular aggregation game: Each individual player $i \in N$ is initially associated with a vector of properties of size 3,

(p_i, q_i, k_i), such that $p_i \geq q_i > 0$ or $p_i = q_i = 0$. Let the null vector of properties $y^0 = (0, 0, 0)$, $D = \{(p, q, k) : k \in \mathcal{N}$ and $(p \geq q > 0,$ or $q = p = 0)\}$, and the aggregation function $g((p_1, q_1, k_1), (p_2, q_2, k_2)) = (p_1 + p_2, \max\{q_1, q_2\}, k_1 + k_2)$. Finally, let $V_1(y^0) = 0$ and $V_1((p, q, k)) = \max\{q, p/k\}$, otherwise. The game (N, V) is not a centralizing game though $V_1(g((p_1, q_1, k_1), (p_2, q_2, k_2)))$ is in between $\min\{V_1(p_i, q_i, k_i) : i = 1, 2\}$ and $\max\{V_1(p_i, q_i, k_i) : i = 1, 2\}$ but it is not strictly increasing in $V_1(p_2, q_2, k_2)$ assuming that (p_1, q_1, k_1) is maintained fixed. To see this, take for example, $(p_1, q_1, k_1) = (5, 3, 2)$ and $(p_2, q_2, k_2) = (6, b, 3)$. Thus, $V_1(5, 3, 2) = 3$, and for $b \geq 2$, $V_1(6, b, 3) = b$. In particular, for $b \in (2, 3)$ the value of $V_1(6, b, 3) = b$ is strictly increasing in b where the value of the aggregated vector, $V_1(g((5, 3, 2), (6, b, 3))) = V_1((11, 3, 5)) = 3$, is not. Thus, the second item in Claim 1 is not satisfied. This means that Theorem 4 cannot be invoked directly to prove that the makespan game with preemptions is totally balanced. Below we use a different approach to prove this assertion.

Claim 4 *The makespan with preemptions game is totally balanced, and a convex subset of its nonnegative core is fully identified.*

Proof:

The characteristic function of the makespan game with preemptions (N, V) can be presented as $V(S) = \max\{U^1(S), U^2(S)\}$ where $U^1(S) = q(S) = \max\{q_i : i \in S\}$, and $U^2(S) = \frac{p(S)}{k(S)} = \frac{\sum_{i \in S} p_i}{\sum_{i \in S} k_i}$.

According to Claim 3, the maximum game (N, U^1) is concave and therefore it is totally balanced and its core is nonnegative and it can be completely characterized.

The form of the characteristic function of the game (N, U^2) is similar to the form of the characteristic function in the queueing game presented in Subsection 5.1 as it is the ratio of a nonnegative real number by a positive number, and it is defined as 0 for the null vector of properties. As shown above, a regular game with this form of a characteristic function is a centralizing aggregation game that has the decreasing differences property. Therefore, according to Theorem 1, the game (N, U^2) is totally balanced, the core of its auxiliary game is fully identified and it coincides with the nonnegative core of the game (N, U^2). The total balancedness of the makespan game with preemptions thus follows from Observation 3.

■

Note that the arithmetic mean game presented in Section 3 is a special case of the game (N, U^2) implying that it satisfies all the conditions required by Theorem 4. Thus, the arithmetic mean game is totally balanced.

6 Conclusions

The determination of whether a given cooperative game has a nonempty core, let alone identifying core cost allocations if the answer is positive, is a challenging task because of the exponential size of the problem, see Section 1. Thus, the identification of general sufficient conditions for proving the total balancedness of cooperative games may greatly simplify this task and unify the research in the field. As mentioned in Section 1, the literature provides just a few sufficient conditions for total balancedness: a concave game where the entire core is fully characterized, a market game where a single core cost allocation can be derived, and regular subadditive and homogenous of degree 1 games that are known to be totally balanced but no general core cost allocation for them has been identified yet. In this article we propose a new sufficient condition: a regular centralizing aggregation game that has the decreasing differences property is totally balanced and, moreover, its non-negative part of the core is fully characterized. For such a game we propose a concave auxiliary game whose core is contained in the core of the original game. Applications in queueing and scheduling games are presented.

An interesting open question deals with cooperative games where each of its players is associated with a score and the characteristic function is one of the various generalized mean functions. We note that the games whose characteristic function is the arithmetic or the geometric means, mentioned in this paper, are two specific such means. The maximum of a collection of scores is also considered as a generalized mean. More specifically, let $(\alpha_1, \ldots, \alpha_n)$ be a vector of positive scores. Let p be a real number. The generalized mean that is associated with $p \neq 0$ is

$$M_p(\alpha_1, \ldots, \alpha_n) = \left(\frac{1}{n} \sum_{i=1}^{n} \alpha_i^p \right)^{\frac{1}{p}},$$

where for $p = 1$, M_p is the arithmetic mean. For $p = 0$ we get the geometric mean:

$$M_0(\alpha_1, \ldots, \alpha_n) = (\Pi_{i=1}^n \alpha_i)^{\frac{1}{n}}.$$

In addition, $M_{-\infty}(\alpha_1, \ldots, \alpha_n) = \min\{\alpha_1, \ldots, \alpha_n\}$ and $M_{\infty}(\alpha_1, \ldots, \alpha_n) = \max\{\alpha_1, \ldots, \alpha_n\}$ are also considered as generalized means. A further generalization deals with the case that the scores do not necessarily have the weight of $\frac{1}{n}$ each. Each player i is then associated with a score α_i and relative weight $w_i' > 0$. For any given coalition S, the real weight for player $i \in S$, is calculated by normalizing the relative weights, i.e., $w_i = \frac{w_i'}{\sum_{j \in S} w_j'}$, implying that $\sum_{i \in S} w_i = 1$. The various generalized means for any coalition take into account these weights. For $p \neq 0$, $M_p(\alpha_1, \ldots, \alpha_n) = (\sum_{i=1}^n w_i \alpha_i^p)^{\frac{1}{p}}$, and for $p = 0$, $M_0(\alpha_1, \ldots, \alpha_n) = \Pi_{i=1}^n \alpha_i^{w_i}$, the weighted geometric mean, is generated.

As we have seen in this article, the arithmetic mean game satisfies the conditions of Theprem 4, and therefore it is totally balanced. The maximum game, on the other hand, is not a centralizing game but it is concave and therefore it is also totally balanced. An interesting question to investigate is whether all the generalized mean games are totally balanced.

Acknowledgment

This research has been supported by the Israel Science Foundation, grant no. 109/12 of the author, and also, it has been partially funded by the Israeli Institute for Business Research.

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