Disagreement, Habit and the Dynamic Relation Between Volume and Prices

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Abstract

Dynamic asset pricing models typically do not generate trading volume whereas empirically trading volume is strongly related to asset returns; volume is usually high when returns are high and during periods of high return volatility. Stock returns on the other hand are known to be quite volatile and offer a high equity premium while the risk-free rate of return is low and quite stable. We attempt to reconcile all these price and volume characteristics in a new model of disagreement where agents have external habit formation preferences that generate time-variation in risk-aversion. The study first shows that both risk-aversion and disagreement have important implications for prices, returns and trading and the relation between them. The model is then calibrated to show that it can account for several characteristics of the stock returns as well as their relation with volume.

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1 Introduction

Despite the long theoretical and empirical literature on the relationship between prices and trading we still lack a unifying theoretical framework by which we can study and understand the principal determinants of both whereas explaining both is essential to understanding the workings and the efficiency of the financial markets. It has been empirically established that periods of high volume coincide with periods of high market returns and high return volatility.[1] It is also known that risky assets are highly volatile in relation to the size of macroeconomic risk and demand a high risk-premium while the risk-free interest rate is known to be quite stable by comparison.[2] The asset pricing literature has been successful in providing possible explanations to the risk-return tradeoff but these models are completely silent when it comes to volume.

With this paper we first provide a unifying and tractable framework with which we study and generate predictions about several asset pricing and trading volume characteristics.[3] Secondly, our point of departure is the asset pricing theory that perceives prices as moving due to changes in the risk-aversion of the economy, which in this paper we generate using external habit formation preferences.[4] We then introduce investor disagreement which is a very promising candidate for explaining the extraordinary trading volume that we observe.[5] Agents receive imprecise information about the growth rate of the economy, that changes every period, and disagree about its interpretation.[6] This disagreement causes agents to hold heterogeneous beliefs about the state of the economy and as a result the find it optimal to engage in speculative trading. The economic patterns and predictions that emerge are very interesting and may prove very informative. The reason, as we will show, is that the time varying risk aversion that is needed to explain prices also affects the amount of trading whereas disagreement that is needed to explain trading also affects prices. With the help of a general equilibrium model of two heterogeneous agents we analyze the pricing and trading effects of the time-varying risk-aversion, disagreement and the time-variation in disagreement. We then analyze their joint effects and show that the model can account for the two main return-volume relations, that is the positive correlations between volume and stock return volatility and between

[1] Interestingly the two observed patterns are reflected in the adages "it takes volume to move prices" and "volume is heavy in bull markets and light in bear markets". Karpoff (1987) reviews the early empirical and theoretical literature on the price-volume relationship and establishes these two empirical regularities. Further studies on the empirical volume-volatility relation are those of Schwert (1989) and Gallant et al. (1992). Other empirical studies that analyze the dynamic relation between trading and returns for either the cross-section or the aggregate stock market include Campbell et al. (1993), Llorente et al. (2002), Chordia et al. (2007) and Griffin et al. (2007) while Lo and Wang (2000) offers a further list of references. Further evidence is provided by Hong and Stein (2007) that show the striking correlation between annual changes to the level of the market and changes in the turnover of the NYSE.

[2] These are the well known equity premium, risk-free rate and volatility puzzles whose literature started with Mehra and Prescott (1985) and Weil (1989) and has been developing since.

[3] Buraschi and Jiltsov (2006) also study both prices and volume and show that a theoretical model of heterogeneous beliefs helps explain the option trading volume and the option price patterns.

[4] For example Campbell and Cochrane (1999) provide a time-varying risk-aversion model using external habit-formation preferences that accounts for the high equity premium, the high asset return volatility, the low risk-free rate, the low volatility in the risk-free rate and the predictability of stock returns.

[5] Disagreement in general refers to the differential interpretation of common information as in Kandel and Pearson (1995).

[6] Veronesi (2000) showed that imprecise information has pricing implications in that it decreases the equity premium.

stock returns and volume, as well as several characteristics of stock returns.

The pricing effects of the assumption of time-varying risk-aversion are well understood in the literature and it stands as one possible explanation of asset pricing facts.[7] Risk-aversion is believed to increase during economic downturns causing an increase in the equity premium, a decrease in stock prices and an increase in stock return volatility.[8] Its trading implications, however, have not been examined. What we show is that time-varying risk-aversion has significant implications on the volume-volatility relation. In particular, we show that risk-aversion affects the amount of trading due to hedging as well as the amount of speculative trading. Starting with the first, we show that in an economy where agents are heterogeneous in their wealth then time-varying risk-aversion generates time-varying hedging needs because less wealthy agents require insurance from the more wealthy when risk-aversion increases. As a result theory predicts that the amount of hedging trading increases in economic downturns.

On the other hand, the effect of the time-varying risk-aversion on the amount of speculative trading is the opposite. We show that given a certain level of belief heterogeneity the amount of speculative turnover is inversely proportional to risk-aversion and hence the amount of speculative turnover should decrease during economic downturns. The net effect obviously depends on the relative size of the two types of trading but in our model the amount of speculative turnover is higher. In fact hedging turnover is an outcome of disagreement and the fact that it generates time-variation in the wealth distribution. As a result time-varying risk-aversion is that it generates negative correlation between volume and return volatility which goes against the empirical relation.

The assumption of disagreement, or also referred to as opinion differences, is known to be able to generate trading.[9] Even though the trading mechanism might vary across models the main result is robust in that the higher the disagreement the higher is the expected amount of speculative trading. Disagreement, however, also has important pricing effects in particular when agents have power utility preferences, with or without external habit. The fundamental effect comes from the pricing effect of belief heterogeneity which is the following: An increase in belief heterogeneity decreases the demand for financial assets and as a result their prices, since their supply is constant.[10] This result is explained by the fact that when agents have heterogeneous beliefs and therefore invest differently, an increase in their holdings of financial assets increases their wealth risk. Therefore, even though agents increase their demand for the specific assets that they believe underpriced the

[7]Time-varying risk-aversion comes from habit formation preferences as in Sundaresan (1989), Constantinides (1990), Ferson and Constantinides (1991), Detemple and Zapatero (1991) and Campbell and Cochrane (1999) among others.

[8]Recently Santos and Veronesi (2010) showed under reasonable assumptions habit-formation preferences have counter-factual implications in that they generate a "growth" rather than a "value" premium.

[9]For the relevant literature see Varian (1989), Harris and Raviv (1993), Scheinkman and Xiong (2003) and Banerjee and Kremer (2010)

[10]Related results and other asset pricing effects of belief heterogeneity can be found in Abel (1990), Detemple and Murthy (1994), Zapatero (1998), Basak (2005), Jouini and Napp (2006b), Jouini and Napp (2006a), Gallmeyer and Hollifield (2008) and David (2008).

net effect on the overall demand for financial assets is in fact negative.[11]. We analyze and calibrate two model setups, one with constant and one with time-varying disagreement.

In a constant disagreement model the principal factor responsible for the variation in trading activity is the level of belief heterogeneity. By the level of belief heterogeneity we mean a weighted average of the differences in individual beliefs where the weights are wealth related. This is because when agents have power utility preferences, that is their risk-aversion is wealth dependent, then the effect of beliefs on asset prices is also wealth dependent. Belief heterogeneity then decreases price levels and increases both the stock return volatility and the amount of speculative turnover. To counteract the effect of time-varying risk-aversion on the volume-volatility relation and generate a significant positive correlation we need to assume a particularly high level of disagreement. However, this has adverse pricing implications in that it predicts excessive volatility in the risk-free rate and the stock price while it also decreases the equity premium by up to 50%. The effect on the equity premium is due to a decrease in the correlation between stock returns and consumption growth.

We then introduce time-variation in disagreement and therefore an additional factor that affects both prices and trading. An increase in disagreement increases the level and the volatility of belief heterogeneity through both wealth reallocation and changes in individual beliefs. Qualitatively, therefore, disagreement has similar effects to those of the level of belief heterogeneity. Specifically, an increase in disagreement increases the expected amount of trading, the return volatility and the risk-free rate while it decreases the stock prices and the equity premium. Quantitatively, however, the effects are more significant and in particular the effect on return volatility.[12] Despite this, a model where disagreement is uncorrelated with the fundamentals of the economy is unable to explain both prices and the return-volume relations.[13] If a high enough unconditional average for disagreement is assumed then the return-volume relations are generated but with adverse pricing predictions similar, even though less pronounced, to the case with constant and high disagreement. We finally show that the model is able to generate the return-volume relations and at the same time explain stock returns well when disagreement is assumed to be sufficiently positively correlated with risk-aversion.

Risk-aversion and disagreement despite being important for price levels, expected returns, stock return volatilities and expected trading volume they have no direct effect on the correlation between stock returns and volume. For example, even though due to disagreement low prices and high expected returns are positively correlated with high expected volume stock returns are not necessarily positively correlated with volume. Statistically, the reason is that volume is much more volatile than the variation in its conditional average. Theoretically, it is because on equilibrium

[11] The effect requires that the utility curvature parameter (the constant relative risk aversion in the absence of habit) is higher than one whereas the effect reverses when it is less than one

[12] A similar result with a similar mechanism is found in Dumas et al. (2009).

[13] Banerjee and Kremer (2010) provide a model where disagreement generates a positive correlation between volume and volatility. However, the mechanism is different while they do not attempt to explain stock returns.

paths the state of the economy and the level of belief heterogeneity changes with volume independently of whether this change brings an increase or a decrease in prices. This explains the fact that despite the particularly strong correlation between returns and volume the theoretical literature has largely neglected it.[14] The asymmetric behavior of trading volume, as we show, depends on the informational structure and the asset structure. With information structure we mean the timing of the information arrival and the way beliefs are formed whereas with asset structure we mean the set of assets that are available for trade.

The informational structure matters because it determines the source of trading at every moment. For example, trading happens either because of a shock to the beliefs or because of a shock to the wealth distribution. Typically models either have only one source or the two sources coincide in timing. We relax this assumption by separating the two which is done by assuming that economy related information arrives in between shocks to the aggregate endowment. As a result in equilibrium agents switch constantly between speculative and hedging positions. The asset structure in turn matters because it determines the way they form their speculative positions and the way the return to their hedging positions. We show for example that an asset structure that resembles one with a set of call options can generate the positive return-volume correlation in a very general setting. In equilibrium the relatively optimistic agents buy assets that pay well on the upside, for example call options, and decrease their holdings of the market portfolio. If subsequently the market goes up they then increase their wealth and bring additional volume because they increase their holdings of the market portfolio. On the other hand, if the market goes down then the relatively pessimistic agents do not need to change substantially their holdings of the market portfolio because they had already done so when the economy related information was received.

The possible explanation that we provide for the return-volume relation, interestingly does not require any kind of trading frictions or market incompleteness. For example Jennings et al. (1981) attributed this asymmetric behavior to the asymmetric costs of long and short positions with a conjecture that as a result prices are less responsive to bad information with less trading. Karpoff (1987) points that this explanation, even though not perfectly rational, is consistent with the fact that a return-volume relation is absent in the futures market. Our explanation if true might be related to short selling constraints in that short selling constraints may push investors to find other means of completing the markets and therefore the explanation does not require that information is asymmetrically incorporated into prices. In general the explanation that we provide is in a setting where volume has no informational content. That is, prices perfectly incorporate all available information and opinions, while volume only reflects the speculative and hedging needs. Of course, it may be that in reality the asymmetric behavior of volume is due to its possible informational content, that is for some reason volume carries information that is good news for the stocks. In this paper, however, we do not examine this route but what we show is the way to have a positive return-volume correlation when volume carries no information.

[14] Hong and Stein (2007) show that the correlation in the annual data is around 0.5. Similar data will be used for the model calibration.

The rest of the paper is structured as follows: Section 2 reviews some of the theories related to trading. The model is presented in Section 3 and the equilibrium is derived in 4. The equilibrium without time-varying disagreement is analyzed qualitatively in Section 5 and the model is calibrated in Section 6. The calibrated model is used to build insight about the relative importance of all the elements and the general behavior of the model. Section 7 introduces time-variability in disagreement and the model is again calibrated and analyzed. Section 8 concludes the study.

2 Theories Related to Trading

Theory under rational expectations views asset prices as reflecting all available information which is uniformly interpreted by everyone and any information whether private or public is reflected instantaneously in asset prices. As a result Milgrom and Stokey (1982) show that no trading is required to incorporate any new information.[15] A similar result is obtained by Judd et al. (2003) that show that in an economy with preference heterogeneity and global completeness agents do not trade beyond the first period.[16] The insight gained from the no-trade theorems have spurred the literature that tries to explain trading. The first stream of research views trading as driven by time-varying liquidity (or hedging) needs that arise due to some form of time-varying uncertainty. In the noise-trading literature the time-varying uncertainty is related to informational noise in which case private information also generates informational trading since the positions are not fully revealing.[17] The noise-trading literature has investigated extensively this line of thought and has generated certain insights as to how rational traders behave under informational uncertainty. However, the main shortcoming is that the resulting patterns of volume and prices depend heavily on the exogenous noise process and the magnitude of the liquidity needs.

A second way of generating trade is by relaxing the assumption of the uniform interpretation of information.[18] There are several conceptual advantages to this investigative route: Firstly, there is ample evidence, both empirical and anecdotal, that indeed economic agents may interpret information differently.[19] Secondly, one needs to assume a certain structure for the disagreement process, that is the informational and belief structure that creates it, and this structure endogenously generates both trading and price effects. Thirdly, evidence shows that information release generates trade which in the context of disagreement is an outcome of both convergence and divergence of

[15] Blume et al. (2006) show that the no-trade theorem fails to hold when markets are incomplete since new arrival of information offers new opportunities for risk-sharing.

[16] Berrada et al. (2007) show that in a continuous-time framework with dynamic completeness, preference heterogeneity results in continuous trading coming from continuously changing hedging needs.

[17] See for example Grossman and Stiglitz (1980), Hellwig (1980), Diamond and Verrecchia (1981), Kyle (1985), Admati and Pfleiderer (1988), Grrundy and McNichols (1989), Foster and Viswanathan (1990), Foster and Viswanathan (1993), Long et al. (1990), Kim and Verrecchia (991a), Kim and Verrecchia (991b), Shalen (1993) and Wang (1994). A more extensive discussion of this literature can be found in Harris and Raviv (1993).

[18] Examples of this literature are Varian (1989), Harris and Raviv (1993), Kandel and Pearson (1995), Hong and Stein (2003), Scheinkman and Xiong (2003), Cao and Yang (2009) and Banerjee and Kremer (2010).

[19] Kandel and Pearson (1995) provide evidence that analysts disagree on the interpretation of public information and also that trading volume is higher during public announcements. Similar evidence provided by Chae (2005) shows that abnormal volume increases significantly upon earnings announcements.

beliefs, where convergence and divergence are endogenously generated by the assumed structure of disagreement. The noise-trading literature on the other hand views public information release as decreasing the informational uncertainty generated by an exogenous process and can generate trade only in the presence of private information as shown by He and Wang (1995). Finally, unlike in several noise-trading models, speculative trading due to disagreement does not require the existence of liquidity trading since it is caused by the differences in beliefs. In fact in our model hedging trading arises from changes to the wealth distribution generated by the speculative trades.

The types of theoretical arguments found in the literature concerning the relation between prices and volume depend on the reason behind trading. The noise-trading literature typically establishes a mechanical connection between volume and prices driven by the exogenous noise process. Blume et al. (1994) relates volume with absolute price changes because volume itself is associated with higher precision in the information revealed and therefore bigger price adjustment. Wang (1994) explains this relation in a model with asymmetric information where the informed investors enjoy private investment opportunities. As a result the uninformed investors are unable to extract all the information from prices and the drop in price required by them to buy the stock is related to the magnitude of informational asymmetry. One related matter is that these models as well as a few others in this literature imply that volume itself contains additional information which in turn implies a causal relation in the direction of volume to prices.[20]

The literature on opinion differences generates a different connection between volume and prices. Harris and Raviv (1993) and Scheinkman and Xiong (2003) propose similar arguments about the positive correlation between volume and volatility using models with risk-neutral agents and short-selling constraints. In both models the most optimistic agent holds the stock and the stock changes hands when the beliefs 'cross'. Consequently more 'crossings' imply more trading and higher volatility. Even though not modeled explicitly in these papers the argument put forth hints on a time varying disagreement. Banerjee and Kremer (2010) assume explicitly a time-varying disagreement and draw the link between volume and volatility. However, none of these papers attempt to explain prices and the mechanism through which disagreement affects either prices or trading is different than in our model. For example, in Banerjee and Kremer (2010) because of the preferences assumed disagreement increases return volatility through an increase in the conditional volatility of the cash-flow expectations. However, in our model this effect is very small compared to the other pricing effects of disagreement and also negligible given the total stock return volatility.

The theoretical model is presented next.

[20]Papers with implications about the informational content of volume include Grrundy and McNichols (1989), Kim and Verrecchia (991a) and Barron and Karpoff (2004). Studies that look at the volume-price causal relation are Hiemstra and Jones (1994), Chen et al. (2001), Lee and Rui (2002) and Chuang et al. (2009) among others.

3 The Model

The model is a Lucas-style infinitely lived endowment economy with a single consumption good and two risk-averse agents. The two agents have different expectations about the growth rate of the economy because they disagree on the interpretation of common information a feature that is commonly known as agreeing to disagree. Due to their disagreement the beliefs of the two agents constantly change and do not converge. Their heterogeneity in expectations induces the two agents to continuously trade with each other in perfectly competitive, complete and frictionless financial markets. The heterogeneity in beliefs, the wealth distribution and the state of the economy determines the level of prices.

3.1 Aggregate Endowment

We assume that time is discrete and infinite, $(t = 0, 1, 2, \ldots)$. Production is not modeled and we assume that the aggregate endowment Y can grow either at a high or low rate:

$$\ln\left(\frac{Y_{t+1}}{Y_t}\right) = \mu + \sigma\epsilon^y_{t+1}, \qquad \text{where} \qquad \epsilon^y_{t+1} = \begin{cases} +1, & \text{w. probability } \pi_t, \\ -1, & \text{o/w.} \end{cases} \tag{1}$$

The conditional mean of the growth rate of the economy is time-varying because the probability of the good state ($\epsilon^y = +1$) changes every period according to an autoregressive process:

$$\pi_{t+1} = \phi_\pi \pi_t + \frac{1-\phi_\pi}{2}\left(1 + \epsilon^\pi_{t+1}\right), \qquad \text{where} \qquad \epsilon^\pi_{t+1} = \begin{cases} +1, & \text{w. probability } 1/2, \\ -1, & \text{o/w.} \end{cases} \tag{2}$$

and $\phi_\pi \in (0, 1)$. Under these assumptions the probability π of high growth is always in the set $(0, 1)$, with unconditional mean of 1/2 and unconditional volatility of $0.5\sqrt{\frac{1-\phi_\pi}{1+\phi_\pi}}$. We will refer to π is the growth state of the economy that determines the conditional mean of the growth rate next period. The growth rate of the economy has unconditional moments of μ and σ, and first lag autocorrelation of $\phi_\pi \frac{1-\phi_\pi}{1+\phi_\pi}$ which is always positive and small. The volatility of the conditional mean of growth is equal to $\sigma\frac{1-\phi_\pi}{1+\phi_\pi}$ and hence the uncertainty about the conditional mean is decreasing in ϕ_π. The true probability of the high growth state is not observed by either of the agents who disagree about the interpretation of information that relates to changes in π.

3.2 Information and Beliefs

The two agents in this model, $i = 1, 2$, have incomplete and common information about the state of the economy. Every period they observe the growth realization but do not observe the probability π or the shocks to the probability, ϵ^π. Therefore, the uncertainty about the growth of the economy does not only come from the one period uncertainty, that is ϵ^y, but also from the uncertainty about the conditional mean. We think of t as measuring quarters and $\ln(Y_{t+1}/Y_t)$ as the quarterly growth rate of the economy that becomes public information by the completion or soon after the completion of a quarter. In between each quarter we assume that the two agents receive information that they believe informative as to how the conditional mean of next period's growth has changed. This is modeled by assuming that in the middle of each period at $t + 0.5$, agents receive two signals $s_{t+0.5} \coloneqq (s^1_{t+0.5}, s^2_{t+0.5})$ that are believed to be informative about ϵ^π_t. The two signals take values of either $+1$ or -1 just like ϵ^π.

In reality in an economy there is constant flow of information that is related to how the economy is growing but the actual growth rate of a quarter is not revealed until the end of the quarter or even later when a central authority receives and processes all relevant information and then makes the announcement. Before the announcement agents receive this constant flow of information and trade on it. This constant flow of information, however, is a different type of information than the announcement of the actual growth rate of the economy. The difference lies on how the two types of information affect the beliefs of agents. This also parallels the situation of a company that publishes every quarter its earning results. Prior to the announcement there is information about the company and each agent forms prior beliefs about the state of the company and next quarter's results. Therefore, how agents form their beliefs prior to announcements and how these beliefs are affected after announcements is the central issue for researchers that want to understand trading and its determinants.[21] As argued by Kandel and Pearson (1995), Kim and Verrecchia (1997) and several others, agents interpret information differently. This has spurred the literature on differences of opinion on which we also build our argument. Possibly due to psychological biases, cultural differences, background information, uncertainty about the structure of the economy and/or other reasons agents interpret information differently and form different prior beliefs.[22] Once an announcement is made or the quantity for which beliefs were formed is realized and a certain uncertainty is resolved all agents update their beliefs but since they had already formed different prior beliefs they also update their beliefs differently. This can possibly explain trading both before and after each announcement or resolution of a certain uncertainty. If this is true and since agent beliefs do not seem to converge it implies that prior to each announcement opinions on average diverge due to the flow of relevant information and upon announcements they converge as for example in Banerjee and Kremer (2010). This is what we will also assume.

In order to capture this realistic feature of the financial economy that can potentially explain trading both prior and after announcements we introduced the signals in the middle of each period that are assumed to be interpreted differently by the two agents and on average cause the divergence of their opinions. We would ideally like to have several such signals in between realizations of the growth rate of the economy or even a continuous flow but in order to keep the model as parsimonious as possible we introduce only one. These two signals are interpreted differently by the two agents by the assumption that they disagree on how informative each of the signals is. We would like to have the following features: (i) the two signals jointly carry some information in the sense that

[21] A very good survey of this literature on trading around announcements is presented by Bamber et al. (2011).

[22] Disagreement is sometimes attributed to overconfidence similar to Daniel et al. (1998) or Gervais and Odean (2001).

they can never be both wrong so that the two agents need to utilize all information, (ii) there is one parameter that controls disagreement and (iii) the two signals are and are perceived to be unconditionally independent and identically distributed. We therefore assume that the beliefs of agent $i = 1, 2$ about the realization of the two signals conditional on the actual realization of ϵ^π is given by:

$$\text{Beliefs } i: \qquad \mathbb{P}^i(s^1, s^2|\epsilon^\pi) = \left(\frac{1}{2} + \rho\right) \frac{\mathbb{1}\{s^i = \epsilon^\pi\}}{2} + \left(\frac{1}{2} - \rho\right) \frac{\mathbb{1}\{s^j = \epsilon^\pi\}}{2}, \qquad j \neq i, \tag{3}$$

where for a certain period s^i refers to $s^i_{t+0.5}$ and ϵ^π to ϵ^π_t. Parameter ρ controls disagreement and we require that $\rho \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. The indicator function $\mathbb{1}\{A\}$ takes the value of one if statement A is true and zero otherwise. The probability measure $\mathbb{P}^i$ represents the beliefs of agent i.[23] This is a parsimonious way of modeling the perceived informativeness of the signals and the disagreement between agents with the features we want to maintain. Notice that the two agents disagree due to ρ. If ρ is equal to zero then there is no disagreement between agents. In this case both agents believe that there is 1/2 probability that both signals will be correct, 1/4 probability that only signal one will be correct and 1/4 probability that only signal two will be correct. We also assume that this is the true distribution and informativeness of the signals, that is:

$$\mathbb{P}(s^1, s^2|\epsilon^\pi) = \frac{1}{4}\mathbb{1}\{s^1 = \epsilon^\pi\} + \frac{1}{4}\mathbb{1}\{s^2 = \epsilon^\pi\}. \tag{4}$$

However, when ρ is different than zero then agents differ in their beliefs about the informativeness of each signal and are boundedly rational in the sense that they do not update their beliefs about the true informativeness of the signals. This is a standard assumption in the differences of opinion literature in order to maintain heterogeneity in beliefs as for example in Scheinkman and Xiong (2003), Dumas et al. (2009), Banerjee and Kremer (2010) and many others. Parameter ρ then determines the amount of disagreement between the two agents in the sense of the following remark:

Remark 1. *The disagreement about the informativeness of each signal s^i is given by,*

$$\left|\mathbb{P}^1(s^i|\epsilon^\pi) - \mathbb{P}^2(s^i|\epsilon^\pi)\right| = |\rho|.$$

where $i \in \{1, 2\}$.

Regardless, however, of the value of ρ each agent believes that there is 1/2 probability that both signals are correct, 1/2 probability that only one of them is correct and zero probability that

[23] Another way of presenting assumption 3 is as follows:

	$\mathbb{P}^1$	$\mathbb{P}^2$
$s^1 = s^2 = \epsilon^\pi$	$1/2$	$1/2$
$s^1 = \epsilon^\pi \neq s^2$	$1/4 + \rho/2$	$1/4 - \rho/2$
$s^1 \neq \epsilon^\pi = s^2$	$1/4 - \rho/2$	$1/4 + \rho/2$
$s^1 = s^2 \neq \epsilon^\pi$	0	0

both are wrong. They differ, however, on the signal that they believe more likely to be correct. Unconditionally the two signals are and are believed by both agents to be independent since each joint realization of the two signals has and is believed to have probability of 1/4 and the realization of each signal has and is believed to have unconditional probability of 1/2. Finally, when interpreting the signals, that is when agents extract the relevant information from the signals about the change in the mean growth rate of the economy they use different weights for the two signals when ρ (disagreement) is different than zero. This is shown in the following lemma:

Lemma 1. *For each agent $i \in \{1, 2\}$ the expectation of ϵ^π given the signals s is given by:*

$$\mathbb{E}^i(\epsilon^\pi | s^1, s^2) = \left(\frac{1}{2} + \rho\right) s^i + \left(\frac{1}{2} - \rho\right) s^j, \qquad j \neq i.$$

Lemma 1 states that an agent's beliefs about the shock ϵ^π is a weighted average of the signals received where the weights depend on ρ and are opposite for the two agents. As a result the two agents interpret the common information differently which causes a divergence in their beliefs when ρ is different than zero and only when the two signals are conflicting.

The common information available to the two agents is the history of the shocks $\epsilon := (\epsilon^y, \epsilon^\rho)$ observed at the beginning of every period, where ϵ^ρ will be the shock to disagreement when we introduce it, and the history of the two signals s observed in the middle of every period. The information set at the beginning of a period t is denoted with $\mathcal{F}_t$ and at the middle of a period with $\mathcal{F}_{t+0.5}$. A fundamental quantity in our theoretical model is the expectation of an agent about the current value of π_t which determines her expectations about the growth rate of the economy. The posterior expectations of agent i about the π_t after the realization of the growth of the economy at time t is denoted with $\pi^i_t := \mathbb{E}^i(\pi_{t-1} | \mathcal{F}_t)$. Once agents observe the new signals at time $t + 0.5$, given their beliefs about the informativeness of the signals and given the law of motion of π form new priors about the new probability π_t of the high growth state according to,

$$\pi^i_{t+0.5} := \mathbb{E}^i(\pi_t | \mathcal{F}_{t+0.5}) = \phi_\pi \pi^i_t + \frac{1 - \phi_\pi}{2} \left[1 + \mathbb{E}^i(\epsilon^\pi_t | \mathcal{F}_{t+0.5})\right], \tag{5}$$

where the conditional expectation of the shock ϵ^π is given by Lemma 1. The quantity π^i_t for agent i is required in order to form expectations about the future as given by the following lemma:

Lemma 2. *Agents form expectations according to the following beliefs about the realization of the next period's shock,*

$$\mathbb{P}^i(s_{t+0.5}, \epsilon_{t+1} | \mathcal{F}_t) = \frac{1}{4} \mathbb{P}^i(\epsilon^y_{t+1} | \mathcal{F}_{t+0.5}) \mathbb{P}(\epsilon^\rho_{t+1} | \epsilon^y_{t+1}),$$

where $\mathbb{P}^i(\epsilon^y_{t+1} | \mathcal{F}_{t+0.5}) = \pi^i_{t+0.5} \mathbb{1}\{\epsilon^y_{t+1} = 1\} + (1 - \pi^i_{t+0.5}) \mathbb{1}\{\epsilon^y_{t+1} = -1\}$ and $\pi^i_{t+0.5}$ is a function of π^i_t, $s_{t+0.5}$ and ρ as given by Lemma 1 and equation (5).

Note that in forming expectations about the future, as given by Lemma 2, we do not require the precisions of the beliefs of the two agents. The precisions are required when the agents form their posterior beliefs about π after observing the realization of endowment growth. In order to keep the model simple we assume that the precision of the prior beliefs of the two agents is the same and constant over time and as a result the posterior expectations about π are assumed to be formed according to

$$\pi^i_{t+1} = \kappa \pi^i_{t+0.5} + (1-\kappa)\mathbb{1}\{\epsilon^y_{t+1} = +1\}, \tag{6}$$

where $\kappa \in (0,1)$ is constant over time. The above updating corresponds to the case where the prior beliefs have a Beta distribution which is known to be the conjugate prior to the binomial distribution. We do not model, however, the beliefs about π since to our knowledge there does not exist a distribution that will remain of the same type over the two stages of change in beliefs within a period. If we assume for example that the posterior beliefs of π_{t-1} are Beta distributed the prior beliefs of π_t after observing the signals $s_{t+0.5}$ will not be Beta distributed any more. What we loose is that we cannot internalize the formation of the posterior beliefs. One solution would be to keep track of the conditional variance of π and use an updating rule based on this, however, this would introduce one more state variable and would complicate both the approximation of the equilibrium returns and the analysis of the model and therefore we opt not to. The precision of the beliefs is the same across agents due to the assumption that their beliefs about the signals are symmetric. The precision over time, however, does vary because the precision of the information coming from the signals depends on ρ, which we will assume later to vary over time, and on whether the signals are conflicting or not. We believe, however, that the predictions of the model and the conclusions of our analysis would not change if we could model completely the beliefs of the two agents.[24]

3.3 Consumption and Preferences

Agents in this economy consume every half period and C^i denotes the consumption process of agent i. We need to introduce consumption in the middle of every period to maintain consumption as the numeraire good. In the absence of consumption in the middle of the periods we would have a free parameter by which we would specify the level of prices. Even though the choice of the free parameter would not affect consumption and the wealth distribution it would affect returns and trading volume. The assumption that will be made make the two approaches identical for a specific choice of this free parameter but for illustration purposes we choose to include consumption every half period.[25]

[24]Veronesi (2000) has shown that precision of information affects asset returns but we do not believe that our predictions would change significantly especially given the fact that asset returns are primary driven by the external habit formation that we will assume.

[25]A more realistic modeling would have continuous production, or several sub-periods of production and hence consumption but agents would not be able to infer the aggregate production of the economy every sub-period. The aggregate production would only be revealed every several sub-periods or every whole period. Such a setup would also allow us to model the continuous flow of information about the state of the economy and hence generate trading even in the absence of hard economic data in most sub-periods as it happens in reality. This approach is required when we assume that most of every day trading volume comes from heterogeneous beliefs and not liquidity needs.

Agents have power utility preferences over streams of consumption in surplus of an external habit similar to Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1991), Campbell and Cochrane (1999) and others:

$$U_i(C^i, X) = \mathbb{E}^i \left\{ \sum_{t=0}^{\infty} \delta^t \left[u(C_t^i, X_t) + \sqrt{\delta} \cdot u(C_{t+0.5}^i, X_t) \right] \middle| \mathcal{F}_0 \right\}, \tag{7}$$

where $\delta \in (0, 1)$ is the subjective discount factor. The subperiod utility is given by

$$u(C, X) = \frac{\left(C - \beta \frac{X}{2}\right)^{1-\gamma} - 1}{1 - \gamma}, \tag{8}$$

where $\gamma > 0$ is the utility curvature parameter and X is the external habit common to both agents. The coefficient of relative risk aversion is state dependent and is equal to $\gamma C/(C - \beta X/2)$. Habit preferences are known to be able to generate several features of the observed asset returns as for example the high equity premium and the high return volatility with a low curvature parameter γ and low real aggregate risk. The external habit can be shut down by setting β to zero in which case the preferences simplify to the standard power utility preferences. As β increases the effect of habit increases since consumption gets closer to habit and the marginal utility of a unit of consumption increases.

The reason for which we introduce habit preferences is two-fold: First we need to match the return volatilities that we observe in the data in order to see whether the relation between volume and volatility can be generated and how. Secondly, since external habit and the variation in risk-aversion that it generates is one of the most important asset pricing theories we would like to investigate the effect it has on trading and the relation between volume and returns. We will show that the combination of variations in risk-aversion generated by habit and variations in the distribution of wealth generated by the heterogeneity in beliefs creates the need for agents to trade for risk-sharing purposes. Further, we will show that variations in risk-aversion generates variations in the level of speculating trading.

In order to keep things simple and comparable to the case where consumption takes place only once every period, we assume that X is constant throughout a given period. As it is typical in the literature, we assume that the external habit is a weighted average of past aggregate consumption, as for example in Ryder and Heal (1973), and it is given by:

$$x_{t+1} = \phi_x x_t + (1 - \phi_x) y_t, \tag{9}$$

where x and y are the natural logarithms of the corresponding variables. In the assumed preferences X is divided by two since consumption is divided over two instances within a period whereas it changes every whole period. Even though it is typical with this form of habit preferences to assume that habit is internal, in order to simplify the model we rather assume it to be external as in the “keeping up with the Joneses” preferences of Gali (1994).

Agents are able to consume in the middle of the period out of what they decide to store in the beginning of the period. Agents can store without cost the consumption good between time t and $t + 0.5$, but not between $t + 0.5$ and $t + 1$. This assumption and the assumption that X is constant between t and $t + 0.5$ are what make this setup identical to a setup where consumption occurs only every whole period. In equilibrium it turns out that each agent consumes what she stores even though the two agents are able to trade both the consumption good and the financial assets in the middle of every period.

3.4 Financial Markets

Between time t and $t+0.5$ and between time $t+0.5$ and $t+1$ there are four and two possible states of nature respectively. Between the beginning of a period and the intermediate period the four states are generated by the two signals s and between the intermediate and the end of a period the states are generated by $\epsilon \coloneqq (\epsilon^y)$. Later when we introduce stochastic variations in disagreement ρ with shocks that are realized every whole period we will also have four states of nature in the second subperiod. Financial markets are assumed to be dynamically complete which means that there are always enough independent assets to span each half-period uncertainty. As a result agents need to trade constantly.[26]

Since markets are complete the asset structure does not have any effect either on equilibrium allocations or on equilibrium prices. The asset structure refers to the type of assets available for trade. However, the asset structure does have an effect on trading volume and its relation to returns. The first asset that is always available for trading is the market which is an infinitely lived stock that pays the aggregate endowment and is in unit net supply. The stock's price is denoted with $\widetilde{P}$ and its price-dividend ratio, which is stationary, is denoted with $P \coloneqq \widetilde{P}/Y$. For the rest of the paper we will refer to P as simply stock price. The rest of the asset structure will be specified later on when we will analyze trading. For deriving the financial equilibrium and the equilibrium stock price we will only require the assumption of dynamically complete markets. The asset structure that we will use will be an example of one that in combination with the information structure we have already assumed can generate the positive correlation between volume and stock returns. We will postpone this discussion for later on.

[26] In a similar setup where there are different shocks and different type of uncertainties every period, as for example with idiosyncratic shocks, but without heterogeneous beliefs there is also the need to constantly trade if markets are only dynamically complete. No trading beyond the first period can be achieved if the financial markets have enough independent assets to span all uncertainty of all periods as shown by Judd et al. (2003). The introduction of heterogeneous beliefs, however, introduces an additional source of uncertainty which is the endogenously varying wealth distribution. In such a case it is difficult to imagine a way to have no trading unless we assume that there exist contingent claims whose payoff depends on the wealth distribution which is obviously not true. For this theoretical reason heterogeneity in beliefs stands as probably the most important determinant of trading activity.

4 Financial Equilibrium

This section presents the financial equilibrium conditions, prices and holdings. The equilibrium is then expressed in a recursive form with respect to the appropriate state vector. The recursive characterization of prices allows the qualitative analysis in terms of the main factors that determine asset prices in equilibrium.

4.1 Equilibrium Conditions

We consider first the optimization problem of each agent. Each agent every half period decides on her portfolio holdings that we denote with the vector $\bar{\theta}^i_t$. Further, every whole period each agent decides how much consumption good to store between time t and $t + 0.5$, that we denote with b^i_t. Agents can store the consumption good without cost from the beginning of every period until the intermediate period which implies that the half period interest rate at the beginning of every period will be zero. In the intermediate period agents are allowed to trade both the consumption good and financial assets. Let further $\widetilde{\mathbf{P}}$ denote the vector of asset prices and $\widetilde{\mathbf{R}}$ the matrix of asset payoffs. The optimization problem of agent i can then be written as follows:

$$\begin{aligned} \max_{(\bar{\theta}^i, b^i)} \quad & U_i(C^i, X) \\ \text{s.t.} \quad & C^i_t + b^i_t + \bar{\theta}^i_t \widetilde{\mathbf{P}}_t \leqslant \bar{\theta}^i_{t-0.5} \widetilde{\mathbf{R}}_t \\ & C^i_{t+0.5} + \bar{\theta}^i_{t+0.5} \widetilde{\mathbf{P}}_{t+0.5} \leqslant \bar{\theta}^i_t \widetilde{\mathbf{R}}_{t+0.5} + b^i_t \end{aligned} \tag{$\mathcal{O}$}$$

Let us now define the notion of equilibrium.

Definition 1. *Let $\mathcal{T} = \{0, 1, 2, \ldots\}$ and $\mathfrak{T} = \{0, 0.5, 1, 1.5, \ldots\}$. A financial market equilibrium is a process of portfolio holdings $\{(\bar{\theta}^1_t, \bar{\theta}^2_t); t \in \mathfrak{T}\}$, a process of consumption storages $\{(b^1_t, b^2_t); t \in \mathcal{T}\}$ and a process of asset prices $\{\widetilde{\mathbf{P}}_t; t \in \mathfrak{T}\}$ such that:*

(i) Processes $\bar{\theta}^i$ and b^i are optimal for each agent i, given optimization problem ($\mathcal{O}$) and the process of asset prices $\widetilde{\mathbf{P}}$;

(ii) Given optimal holdings $(\bar{\theta}^1, \bar{\theta}^2)$ financial markets clear for all $t \in \mathfrak{T}$. That is all holdings of the stock sum to one and all holdings of the contingent claims sum to zero.

Speculative bubbles and Ponzi schemes are excluded from this equilibrium even though we have not assumed explicitly the required conditions. A financial equilibrium is well known to generically deliver equilibria with efficient allocations to the extent that markets are always complete in which case the equilibrium is equivalent to a social planner equilibrium with stochastic weights.

Given a set of prices the optimization problem yields the following usual Euler conditions with respect to the holdings of an asset j:

$$\widetilde{P}_t^j = \sqrt{\delta}\mathbb{E}_t^i\left[\left(\frac{2C_{t+0.5}^i - \beta X_t}{2C_t^i - \beta X_t}\right)^{-\gamma}\widetilde{R}_{t+0.5}^j\right] \tag{10}$$

$$\widetilde{P}_{t+0.5}^j = \sqrt{\delta}\mathbb{E}_{t+0.5}^i\left[\left(\frac{2C_{t+1}^i - \beta X_{t+1}}{2C_{t+0.5}^i - \beta X_t}\right)\widetilde{R}_{t+1}^j\right], \tag{11}$$

where $\mathbb{E}_t^i$ is shorthand for $\mathbb{E}^i(\cdot|\mathcal{F}_t)$ (and similarly $\mathbb{P}_t^i \coloneqq \mathbb{P}^i(\cdot|\mathcal{F}_t)$). The above Euler conditions hold at all times, for all assets and for all agents. We note that the expectations are different for the two agents when they hold different beliefs about the probabilities of the future states. The expectations, however, in equation (10) are the same for both agents since the two agents have uniform beliefs about the signals s. This together with the market completeness implies that the marginal rates of substitutions between any whole period t and $t + 0.5$ are the same for the two agents. Therefore, it means that the two agents have the same growth in their consumption surpluses between t and $t + 0.5$. The consumption surplus of an agent refers to her consumption net of her habit. The first order condition with respect to the decision to store b_t is given by:

$$1 = \sqrt{\delta}\mathbb{E}_t^i\left[\left(\frac{2C_{t+0.5}^i - \beta X_t}{2C_t^i - \beta X_t}\right)^{-\gamma}\right]. \tag{12}$$

The above first order conditions 10, 11 and 12 together with the market clearing condition that consumption in the middle of the period is equal to the amount stored, $C_{t+0.5} = b_t^1 + b_t^2$, yield the following results:[27]

Lemma 3. *In equilibrium agents in the middle of the period consume what they stored at time t, that is $C_{t+0.5}^i = b_t^i$. Also, the stochastic discount factor between t and $t + 0.5$ is one for all states and therefore, prices at time t are equal the expectation of their payoffs at time $t + 0.5$, that is $\widetilde{P}_t^j = \mathbb{E}_t\left[\widetilde{R}_{t+0.5}^j\right]$. Finally the following is true:*

$$C_t^i - \beta\frac{X_t}{2} = \frac{C_t^i + b_t^i - \beta X_t}{1 + \delta^{1/2\gamma}}.$$

Lemma 3 verifies that our modeling setup with consumption in the intermediate period is equivalent to a case where there is consumption only every whole period since $C_t^i + b_t^i$ is the total consumption of agent i out of the production Y_t. Further, the stochastic discount factor between time t and $t + 0.5$ is always one because there is no consumption risk and agents are able to equalize their marginal utilities. Consumption risk would be produced if agents held different beliefs about the set of signals s. In such a case the agents would want to speculate and would introduce endogenous consumption variation across states.[28]

[27]The equilibrium condition $C_{t+0.5} = b_t^1 + b_t^2$ is implied by the financial market clearing condition and the fact that at equilibrium the budget constraints of the two agents are satisfied with equality.

[28]This is related to the extraneous risk whose pricing implications were examined by Basak (2000).

We now define a new variable, which is stationary in equilibrium, and it is each agent's share of the total consumption surplus $Y_t - \beta X_t$:[29]

$$\alpha_t^i \coloneqq \frac{C_t^i + C_{t+0.5}^i - \beta X_t}{Y_t - 2\beta X_t} \tag{13}$$

and remember that $C_{t+0.5}^i = b_t^i$. The consumption surplus share of the first agent will be used as a state variable of the economy for reasons that will be explained later. Note that $\alpha_t^1 + \alpha_t^2 = 1$ in equilibrium and also note also that α^i becomes the consumption share in the absence of habit ($\beta = 0$). The last state variable that we need to define relates to the consumption surplus for the whole economy and it is the logarithm of the habit normalized by the level of the aggregate endowment:

$$\omega_t \coloneqq \ln\left(\frac{X_t}{Y_t}\right). \tag{14}$$

It is necessary to normalize the level of the habit since it is non-stationary in this growing economy while ω is, and it will be referred to as the *habit-endowment ratio.* Given its definition and the law of motion of x assumed in (9) the law of motion of the habit-endowment ratio is given by:

$$\omega_{t+1} = \phi_x \omega_t - (y_{t+1} - y_t). \tag{15}$$

The result that the stochastic discount factor is one between t and $t + 0.5$ implies the following pricing condition:

Lemma 4. *In equilibrium the stock satisfies the following Euler equation for each agent:*

$$P_\tau = \delta \mathbb{E}_\tau^i \left[\left(\frac{Y_{t+1}}{Y_t}\right)^{1-\gamma} \left(\frac{1 - 2\beta \exp(\omega_{t+1})}{1 - 2\beta \exp(\omega_t)}\right)^{-\gamma} \left(\frac{\alpha_{t+1}^i}{\alpha_t^i}\right)^{-\gamma} (P_{t+1} + 1) \right], \qquad \tau = t, t + 0.5.$$

Lemma 4 shows that the price of the stock satisfies the usual condition and the price changes in the middle of a period due to changes in expectations. The stochastic discount factor in Lemma 4 is comprised of three parts.[30] The first part is the standard stochastic discount factor of a homogeneous agent economy with standard power utility preferences. The second part is introduced because of the external habit and it vanishes when the parameter β is set to zero. The last part arises when

[29]The stationarity of this variable, which is a proxy of the wealth held by an agent, is an outcome of the assumption that every agent on average is equally wrong with respect to the true probabilities of the states of nature. This means that no agent has superior beliefs so that in time she ends up with the entire wealth of the economy.

[30]Note that the stochastic discount factor for agent i is equal to

$$\left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma} \left(\frac{1 - 2\beta \exp(\omega_{t+1})}{1 - 2\beta \exp(\omega_t)}\right)^{-\gamma} \left(\frac{\alpha_{t+1}^i}{\alpha_t^i}\right)^{-\gamma}$$

since the pricing condition of Lemma 4 is written for the price of the stock normalized by the aggregate endowment, the price dividend ratio.

agents have heterogeneous consumption processes due to a source of uncertainty that is not the fundamental, which in this model comes from the heterogeneity of beliefs.

4.2 Recursive Characterization of Prices and Holdings

The financial equilibrium can be written in a recursive form with a state vector that fully characterizes the equilibrium quantities. Lemma 4 indicates that the price-dividend ratio of the stock is a function of the individual beliefs, the habit-endowment ratio and the endogenous consumption surplus distribution. Let the share of the consumption surplus of the first agent be denoted with α and hence of the second agent is $1 - \alpha$. The beliefs and their evolution are characterized by the individual quantities π^i and the disagreement ρ. It remains to verify that the state vector $z \coloneqq (\alpha, \pi^1, \pi^2, \omega)$ fully characterizes the financial equilibrium.

The pricing condition of Lemma 4 implies that the marginal rates of substitution multiplied by the probability of the corresponding state is equalized between the two agents. This together with Lemma 2 yield the next result.

Lemma 5. *The law of motion for the consumption share of the first agent is given by*

$$\alpha' = \alpha \left(\frac{\mathbb{P}^1(\epsilon^y|z,s)}{q(\epsilon^y|z,s)} \right)^{1/\gamma}$$

where the prime indicates next period's value and quantity q is defined according to

$$q(\epsilon^y|z,s) \coloneqq \left[\alpha \mathbb{P}^1(\epsilon^y|z,s)^{1/\gamma} + (1-\alpha)\mathbb{P}^2(\epsilon^y|z,s)^{1/\gamma} \right]^{\gamma}.$$

Quantity q is the generalized weighted average of the individual beliefs with exponent the curvature parameter γ and weights the individual shares of consumption surplus. Hence the consumption surplus shares change to the extent that agents have heterogeneous beliefs.

Lemma 5 together with the law of motion of the individual beliefs π^i and the law of motion of the habit-endowment ratio constitute the law of motion of the state vector z which can be generally represented by $z' = L(z, s, \epsilon)$. As a result P and other equilibrium quantities can now be expressed as functions of the state vector. Lemma 4 expresses the equilibrium stock price in terms of the individual expectations. It is convenient to express it in terms of a single probability measure. A natural choice is to use the consumption surplus weighted average of the probability measures of the two agents which can be considered as an unbiased estimator of the true probability measure. For this reason the following quantity is defined:

$$\xi(\epsilon^y|z,s) \coloneqq \frac{q(\epsilon^y|z,s)}{\mathbb{P}^*(\epsilon|z,s)}, \tag{16}$$

where $\mathbb{P}^*(\epsilon|z,s) \coloneqq \alpha\mathbb{P}^1(\epsilon|z,s) + (1-\alpha)\mathbb{P}^2(\epsilon|z,s)$ is the probability measure that we will use. The stock price can now be expressed as follows:

$$P(z) = \delta \mathbb{E}^* \left[J(z, z') \left(\widetilde{P}(z') + 1 \right) \middle| \, z \right], \tag{17}$$

where

$$J(z, z') \coloneqq \xi(\epsilon^y | z, s) \left(\frac{Y'}{Y} \right)^{1-\gamma} \left(\frac{1 - 2\beta e^{\omega'}}{1 - 2\beta e^{\omega}} \right)^{-\gamma}.$$

In the homogeneous agent economy, in which case ξ is identically one, the prices vary due to the variation in the habit-endowment ratio and the variation in the expectation of the aggregate consumption growth. Quantity ξ and how it is affected by the differences in beliefs as well as how it affects the dynamic behavior of prices will be analyzed in the next section.

The two agents need to trade in order to finance their optimal consumption within a period and their new optimal holdings. We denote the equilibrium financial wealth of the first agent standardized by the period's aggregate consumption with W. From the equilibrium pricing and allocation conditions already derived we obtain the following result:

Lemma 6. *The equilibrium financial wealth for the first agent in state z satisfies the following condition*

$$W(z) = c(z) + \delta \mathbb{E}^* \left[J(z, z') W(z') \middle| \, z \right],$$

and once the signals s are observed it is given by

$$W(z, s) = \delta \mathbb{E}^* \left[J(z, z') W(z') \middle| \, z, s \right].$$

The whole period financial wealth is required to finance both the new optimal holdings, the consumption and the storage, that is the total consumption of the period where $c(z) = \alpha + \beta e^{\omega}(1 - 2\alpha)$ is the consumption share. The half period financial wealth on the other hand is only required to finance the new holdings since the consumption storage is what is used for the middle of the period consumption. This accounts for the fact that in Lemma 6 only the whole period financial wealth includes consumption.

The equilibrium wealth function of the first agent W, together with the equilibrium prices can give us the holdings of the two agents in equilibrium since the optimal holdings are constructed so that they generate the optimal financial wealth process for each agent. Let us denote with $\mathbf{R}(z)$ the payoff matrix of all assets from state z to (z, s) and with $\mathbf{R}(z, s)$ the payoff matrix of all assets from (z, s) to $L(z, s, \epsilon)$. Let also $\bar{\theta}(z)$ and $\bar{\theta}(z, s)$ denote the holdings of the first agent in states z and (z, s) respectively. Finally, let $\mathbf{W}(z)$ denote the vector of the first agent's wealth in the four states (z, s) following state z and similarly $\mathbf{W}(z, s)$ for the states from (z, s) to $L(z, s, \epsilon)$. Then, the equilibrium holdings and the resulting trading volume is obtained by,

$$\bar{\theta}(z) = \mathbf{R}(z)^{-1} \mathbf{W}(z) \qquad \text{and} \qquad \bar{\theta}(z, s) = \mathbf{R}(z, s)^{-1} \mathbf{W}(z, s). \tag{18}$$

The holdings of the second agent are obtained from market clearing. We have already derived the equilibrium conditions for W which means that what is left is to specify the assets available for trading and derive their pricing conditions if needed. However, the only long lived asset that we will consider is the stock for which we have already derived its pricing condition. The rest of the assets that will be considered are short-lived half period assets whose payoffs will be exogenously specified.

4.3 Computation of Equilibrium

The model presented in this paper is very tractable in the sense that equilibrium prices and allocations can be computed from a single function to be determined which is $W(z)$, the equilibrium financial wealth of the first agent standardized by the aggregate consumption of the corresponding period. This was achieved first due to the choice of α as a state variable whose law of motion was derived in Lemma 5 by the assumption of market completeness. This means that we need to determine function W in terms of the state vector z whose law of motion is known. The second element that allows us to only need one function to obtain both prices and allocations is the fact that we choose an asset structure that includes the stock and other short-lived assets and the fact that the stock price in this equilibrium can be constructed from the wealth function as shown next.

Since there is no labor income in this economy the equilibrium price of the stock is determined by aggregating the wealth of the two agents plus the aggregate endowment of the period. Hence, the price-dividend ratio of the stock is given by

$$P(z) = W(z) + W(\hat{z}) - 1, \tag{19}$$

where $\hat{z} \coloneqq (1 - \alpha, \pi_2, \pi_1, \omega)$. Due to the symmetrical property of the model with respect to the two agents the equilibrium wealth of the second agent is given by the function W after changing the consumption surplus share and switching beliefs. Once the equilibrium price is obtained the holdings of the two agents can be computed for the asset structure considered in (23).

The only thing needed therefore in order to compute the equilibrium quantities, that is consumption, holdings, wealth distribution and the stock price, is to be able to compute the wealth function W for each state z. The unknown function has no general closed form expression and therefore it needs to be approximated. We approximate it with a complete Chebyshev polynomial of order n.[31] We estimated the values of the polynomial parameters with a projection method applied on the first dynamic functional equation presented in Lemma 6.[32] $\widetilde{W}(z, s)$ is computed using the approximated function $\widetilde{W}(z)$ and the second equation of Lemma 6.

[31]For the results presented in this paper n was chosen to be between 8 and 12 depending on the size of the state vector.

[32]The projection method that was used is a variant of the projection methods presented in Judd (1998).

5 The Dynamic Behavior of Prices, Returns and Trading

In this section the behavior of prices, returns and trading volume will be examined qualitatively in the absence of time varying disagreement. First, the stock price and it's determining factors will be analyzed and then the same will be done with the trading volume once we specify an asset structure. Once the two are analyzed independently we will combine our findings to discuss the relation between volume and returns. In order to facilitate our analysis we will derive approximate expressions for the one-period continuously compounded risk-free rate r^f, the one-period price of risk and the turnover of the stock over a single period which we will denote with T. The approximations are derived in Appendix B.

The factors that affect both prices and volume are (i) the $\mathbb{P}^*$-probability of the high growth state $\pi(z)$ which determines the conditional mean $\mu(z)$ and the conditional volatility $\sigma(z)$ of the log endowment growth, (ii) the differences in beliefs denoted with $\Delta(z) \coloneqq \pi^1 - \pi^2$, (iii) the dispersion in consumption surplus denoted with $h(z) \coloneqq \alpha(1-\alpha)$, which can be regarded as a proxy for the wealth distribution, and (iv) the habit-consumption surplus ratio ω.

5.1 Prices

The behavior of the stock price depends principally on the dynamic behavior of the one-period risk-free rate and the one-period price of risk. Starting from the case with no disagreement, that is $\rho = 0$, the one-period stochastic discount factor is given by

$$M(z, z') = \delta \left(\frac{e^{y'-y} - 2\beta e^{\phi_x \omega}}{1 - 2\beta e^{\omega}} \right)^{-\gamma},$$

which can be approximately expressed in a log-normal form where the risk-free rate is given by

$$r^f(z) \approx -\log(\delta) + \gamma(\omega)\mu(z) + (\gamma(\omega) - \gamma)(1 - \phi_x)\omega - \frac{1}{2}\gamma(\omega)^2\sigma(z)^2 \left(1 + \frac{1}{\gamma} - \frac{1}{\gamma(\omega)}\right) \tag{20}$$

and the time-varying price of risk is approximately equal to $\gamma(\omega)\sigma(z)$ where

$$\gamma(\omega) \coloneqq \frac{\gamma}{1 - 2\beta e^{\omega}} \tag{21}$$

denotes the time-varying risk-aversion. In the above approximated expressions the change in the risk-free rate and the risk-aversion in the middle of the period are not taken into consideration but this has a secondary effect as confirmed by the numerical analysis.

When the parameter β is positive the main driving force for prices is the habit-endowment ratio through the time-varying risk-aversion and its effect on the stock price is increasing in its persistence, ϕ_x. When habit increases relative to aggregate consumption, that is ω increases, the risk-aversion of the economy increases and so does the price of risk. An increase in ω also increases the need to smooth consumption inter-temporally by borrowing against the future which results into an increase in the risk-free rate. The increase in risk-aversion also increases the demand for precautionary savings and therefore placing a downward pressure on the risk-free rate but the net effect on the risk-free rate is positive.

The stock price-dividend ratio decreases with ω both due to the increase in the risk-free rate as well as the increase in the price of risk. As a result the stock return and the endowment growth become strongly positively correlated because ω decreases with endowment growth.

The conditional volatility of the stock return is potentially highly stochastic depending on the habit parameter β. The resulting effect is that when habit increases in relation to consumption the stock return volatility increases. This effect comes from both the risk-free rate and the price of risk through the stochastic relative risk-aversion $\gamma(\omega)$. Due to its form the first derivative of the stochastic risk-aversion with respect to ω is positive and proportional to $\gamma(\omega)$. This means that when ω increases both the risk-free rate as well as the price of risk become increasingly sensitive to any changes in ω and hence the stock price becomes more volatile.

The introduction of belief heterogeneity brings two additional factors to the one period stochastic discount factor through ξ, as defined in (16), which are the difference in beliefs $\Delta(z) = \pi^1 - \pi^2$ and a measure of the wealth dispersion which is given by $h(z) = \alpha(1-\alpha)$. However, both are related to the belief heterogeneity since the impact that an agent has on prices is wealth dependent. Essentially, the product $h(z)\Delta(z)$ can be considered to be an overall measure of the level of belief heterogeneity in the economy. However, we need to examine how each component behaves separately. The difference in beliefs changes every half period and in particular to

$$\Delta(z, s) = \phi_\pi \Delta(z) + (1 - \phi_\pi)\rho(s^1 - s^2)$$

when the signals s are revealed and to $\kappa\Delta(z, s)$ when the endowment growth is realized. It is evident that the difference in beliefs increases only when the signals are different and the increase depends on the level of disagreement as given by ρ. At the end of each period the uncertainty is resolved which causes some convergence in beliefs depending on the level of uncertainty which is given by the parameter κ.

The variable h, which is regarded as a proxy for the dispersion in wealth, inherits its properties from α, the consumption surplus share, whose law of motion is given in Lemma 5. From its law of motion it can be inferred that α changes to the extent that there are differences in beliefs and the bigger the difference in beliefs the bigger the change. Also the magnitude of the changes in α depend on the curvature parameter γ. That is, a higher risk-aversion implies that agents speculate less and therefore their wealth and hence the wealth dispersion is less volatile over time. Quantity h is persistent and its persistence decreases with its level, that is when the wealth is concentrated then it changes more slowly over time. Finally, its conditional volatility is increasing in Δ and decreasing in γ.

In order to understand the pricing effect of belief heterogeneity we need to first see the effect of h and Δ on the one-period interest rate and the one-period price of risk through ξ. It turns out that the effects depend heavily on the curvature parameter γ as shown in the following remark.

Remark 2. *Consider $\alpha \in (0,1)$, $\pi \in (0,1)$ and $\Delta \in \left(0, \min\left[\frac{\pi}{\alpha}, \frac{1-\pi}{1-\alpha}\right]\right)$ as the independent variables and let the beliefs of the two agents for the high growth state be $\pi_1 = \pi + h\Delta/\alpha$ and $\pi_2 = \pi - h\Delta/(1-\alpha)$ where $h = \alpha(1-\alpha)$. Then ξ as defined in (16) has the following properties:*

$$(i)\ \xi \lesseqgtr 1, \qquad (ii)\ \frac{\partial \xi}{\partial (h\Delta)} \lesseqgtr 0 \qquad \textit{and} \qquad (iii)\ \frac{\partial \xi}{\partial \pi} \gtreqless 0 \qquad \textit{when} \qquad \gamma \gtreqless 1.$$

The first two properties of the above remark affect only the equilibrium interest rate. The first property affects ξ independently of the state whereas the second property in this model affects both states equally since there are only two states and Δ has the opposite sign for the two states. The third property affects the risk-premia because it introduces variation across states.

As already noted, ξ will take values below or above one depending on the curvature parameter γ (property (i) of Remark 2). In the most typical case where the two agents have a curvature parameter higher than one prices become discounted in the sense that the higher the level of belief heterogeneity, as measured by $h\Delta$, the lower are the prices (property (ii) of Remark 2). The reason for this can be explained in the following way: If we start from a case of homogeneous beliefs then there are two effects on prices when heterogeneity of beliefs is introduced. The first is that the state prices increase because agents push up the prices of the states for which they believe that their probability is high. For example the optimists push up the prices of the good states of nature while the pessimist the prices of the bad states of nature. The second effect has to do with the fact that agents increase the riskiness of their wealth because they invest differently. They increase their exposure to the source of uncertainty for which they hold different beliefs but with opposite signs. But if agents are risk-averse then the belief heterogeneity decreases their overall demand for investment. But since assets are in a constant supply prices go down. The two effects cancel each other out when γ is equal to one. The riskiness effect is greater when the curvature parameter is higher than one. In this case the price-dividend ratio decreases both with Δ and with h, that is prices decrease with the level of belief heterogeneity. The effect reverses when γ is less than one.[33]

The third property of ξ shown in Remark 2 has an effect on the price of risk. For the most typical case of $\gamma > 1$ the third property states that the lower the probability of a certain state is, the higher is the discounting of that particular state. As a result the heterogeneity of beliefs introduces a component to the price of risk that moves in the opposite direction to $\mu(z)$. It turns out, however, that this effect is quite small and in particular dwarfed by the overall effect of habit.

[33] Earlier research on the asset pricing effects of belief heterogeneity, for example Miller (1977), Harrison and Kreps (1978) and Scheinkman and Xiong (2003), proposed that belief heterogeneity causes a "bubble" effect on prices. It is shown here that the result obtained by these papers is due to assuming risk-neutrality and not an outcome of the short-selling constraint as conjectured. The short-selling constraint is needed when agents are risk-neutral in order to prevent agents from taking infinite positions.

The introduction of belief heterogeneity, therefore, has little effect on the price of risk but it has an effect on the interest rate. The additional element to the risk-free rate as approximated in (20), that shows the effect of belief heterogeneity, can be approximately given by the following expression:[34]

$$2h\frac{\sigma^2}{\sigma(z)^2}\frac{\gamma-1}{\gamma}\mathbb{E}\left[\Delta(z,s)^2|z\right]$$

where $\mathbb{E}\left[\Delta(z,s)^2\right] = \left[(\phi_\pi\Delta(z))^2 + 2(1-\phi_\pi)^2\rho^2\right]$. As already shown in Remark 2 the effect exists as long as γ is different than one, it is positive when $\gamma > 1$ and negative when $\gamma < 1$. Further, the magnitude of the effect on the interest rate is increasing in the level of belief heterogeneity as given by $h\Delta$.

Belief heterogeneity, as already noted, affects the stock price through the effect on interest rates. Being a claim to a stream of future cash-flows the price of the stock is more sensitive to the wealth dispersion, as approximated by h, than to the difference in beliefs as given by Δ, since h is more persistent. In fact h has possibly a strong effect on the level as well as on the volatility of the stock price. For this reason belief heterogeneity has one more effect on the stock through the changes in h. Even though the price of risk is hardly affected by the heterogeneity in beliefs the risk-premium of the stock can be affected significantly. The reason is that the conditional correlation of the stock return with the stochastic discount factor changes over time and it typically decreases with the level of heterogeneity. This is because the conditional correlation between changes in h and the endowment growth depends on the state of the economy. For example in a state where the relatively pessimistic agent is also the more wealthy the conditional correlation between the stock return and the stochastic discount factor is low. This is because a high growth state will lead to an increase in the wealth dispersion and hence a decrease in the stock price.

5.2 Trading

Agents in this economy trade for two reasons. The principal reason is speculation coming from the difference in beliefs while the second reason is hedging due to the change in the state of the economy ω and only when agents are heterogeneous in their wealth. In a given period agents trade twice, the first time when they receive the economy related information s and then again at the end of the period when the period uncertainty is resolved, that is when the endowment growth is realized. In the intermediate periods agents trade for both speculative and hedging purposes whereas the trading at the end of every period is done mostly for hedging purposes. This is because at the end of every period, which is the beginning of a new period, agents hold the same beliefs about the next intermediate period. For this reason we will be referring to the positions held at the beginning of a period as the hedging positions whereas the positions held after s is realized as the speculative positions.

[34] Here we only show an approximate expression of the additional element arising from the introduction of belief heterogeneity.

There are two things on which we need to focus our analysis in order to examine the two price-volume relations. The first that will help us understand the volume-volatility relation is the total amount of trading volume in a given period, in particular the average trading volume, and how this varies with the state of the economy. We will be measuring trading volume with the total turnover of the stock in the intermediate and in the end of a period and it will be denoted with T. The second will help us understand the return-volume relation and it is the dependence of the trading volume in a given period on the realization of the shocks. But first we need to specify the asset structure, that is the additional assets that complete the market.

The assets that dynamically complete the market are half-period zero net supply contingent claims that pay the aggregate endowment of the period only in one of the immediate future states. We also assume that there do not exist contingent claims that have a positive payoff in the state $s_{t+0.5} = (-1, -1)$ between t and $t + 0.5$ and in the state $\epsilon^y_{t+1} = -1$ between $t + 0.5$ and $t + 1$. In the following matrix a given column gives the payoff of the corresponding asset in the four immediate future states and a given row gives the payoff of all the assets in the corresponding future state where all payoffs are divided by the aggregate endowment of the corresponding state:

$$\mathbf{R}_t = \begin{pmatrix} P_{t+0.5}(s_{t+0.5} = (+1,+1)) & 1 & 0 & 0 \\ P_{t+0.5}(s_{t+0.5} = (+1,-1)) & 0 & 1 & 0 \\ P_{t+0.5}(s_{t+0.5} = (-1,+1)) & 0 & 0 & 1 \\ P_{t+0.5}(s_{t+0.5} = (-1,-1)) & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{R}_{t+0.5} = \begin{pmatrix} P_{t+1}(\epsilon_{t+1} = +1) + 1 & 1 \\ P_{t+1}(\epsilon_{t+1} = -1) + 1 & 0 \end{pmatrix}. \tag{22}$$

An asset structure that like the one assumed by (22) that assumes there is one state for which only the stock has positive payoff the investment on the stock is required to generate the wealth of that state. Let in general s^* and ϵ^* denote these states for which only the stock has positive payoff which for the assumed asset structure (22) those states are $s^* = (-1, -1)$ and $\epsilon^* = -1$. Therefore, the optimal holding of the stock by the first agent, which is denoted with θ, is given by

$$\theta(z) = \frac{W(z, s^*)}{P(z, s^*)} \qquad \text{and} \qquad \theta(z, s) = \frac{W(L(z, s, \epsilon^*))}{P(L(z, s, \epsilon^*)) + 1}, \tag{23}$$

for the two subperiods. We will now derive some expressions for the stock turnover.

Let $\vartheta(z)$ denote the fraction of total wealth owned by the first agent in state z. In equilibrium the wealth share of the first agent satisfies the following relation:

$$\vartheta(z) = c(z) + \lambda(z)P(z)$$

where $c(z)$ is the consumption share of the first agent in the period starting with state z and is equal to $\alpha(1 - 2\beta e^{\omega}) + \beta e^{\omega}$. The quantity $\lambda(z)$ is defined by

$$\lambda(z) \coloneqq \mathbb{E}_z \left[\vartheta(z') - \vartheta(z)\right] + \mathbb{C}_z \left[\vartheta(z'), J(z, z')r(z, z')\right] \tag{24}$$

where $\mathbb{E}_z$ and $\mathbb{C}_z$ denote the conditional expectation and conditional variance respectively and $r(z, z')$ the gross stock return. We also denote with $\lambda(z, s)$ the same expectation but after signals s are observed. By construction λ is on average zero while it is generally small. In the special case where $\beta = 0$ and $\gamma = 1$ then λ is always zero. We could expand more on how λ varies and how it affects the equilibrium holdings and trading volume but for length considerations and since the main intuition goes through without it we concentrate on the consumption share $c(z)$.

We start with the equilibrium holding of the stock at the beginning of a period where the state of the economy is z, then given the defined quantities, it is given by:

$$\theta(z) = \vartheta(z) + \lambda(z, s^*).$$

The optimal stock holding, therefore, is largely given by the wealth share of the agent and is little affected by the state s^* which is the state in which only the stock has a positive payoff. At the beginning of every period agents agree on the probabilities of s and as a result they take almost no speculative positions but hold the stock in about the same proportion as the proportion of their wealth.

The stock holding in the intermediate period is formed when the signals s arrive and agents form their new beliefs. It is then given by

$$\theta(z, s) = \vartheta\left(L(z, s, \epsilon^*)\right),$$

that is the first agent's holding of the stock reflects how much of the total wealth the agent wants to have if ϵ^* realizes. Therefore, agents want to change their wealth share from the previous period only to the extent that they have different beliefs and for this reason we consider the positions taken in the intermediate period as speculative. In the case where $\{\epsilon^y = -1\} \in \epsilon^*$ the turnover in the middle of a period can be approximated by the following relation:

$$\begin{aligned} T(z, s) &= \left|\vartheta\left(L(z, s, \epsilon^*)\right) - \vartheta(z) - \lambda(z, s^*)\right| \approx \left|c\left(L(z, s, \epsilon^*)\right) - c(z)\right| \\ &\approx \left|-\frac{h(z)\Delta(z, s)}{\gamma(\omega)} \frac{f_1(z, s)}{1 - \pi(z, s)} + \beta(e^{\phi_x \omega - \mu + \sigma} - e^{\omega})(1 - 2\alpha)\right| \end{aligned}$$

where $f_1(z, s)$ is an expression derived in Appendix B which fluctuates around one.[35] The turnover in the middle of a period has two parts, the first, which is the speculative part, is proportional to the belief heterogeneity $h(z)\Delta(z, s)$ and inversely proportional to the risk-aversion $\gamma(\omega)$. The second part is related to hedging and it comes from the habit preferences and the heterogeneity in wealth. When agents are heterogeneous in their wealth they are also heterogeneous in their risk aversion. Consequently it is optimal for the two agents to have a transfer of wealth from the more wealthy (and less risk-averse) to the less wealthy (and more risk-averse) when the habit-endowment

[35]In the case where $\{\epsilon^y = +1\} \in \epsilon^*$ the difference in beliefs $\Delta(z, s)$ changes sign, the probability in the denominator becomes $\pi(z, s)$ and σ changes sign.

ratio increases, and the opposite when it decreases. As a result the more wealthy provides certain insurance to the less wealthy and in return increases her wealth when the economy does well. The amount of hedging increases with the wealth heterogeneity and with ω that is when consumption is closer to habit.

At the end of the period when the endowment growth realizes, agents incur their speculative losses or gains and revert back to their hedging positions holding the stock in about the same proportion as their new wealth share. As a result the amount of stock turnover is given by

$$T(z, s, \epsilon) = \left| \vartheta\left(L(z, s, \epsilon)\right) - \vartheta\left(L(z, s, \epsilon^*)\right) + \lambda\left(L(z, s, \epsilon), s^*\right) \right|.$$

The above expression has two important points. The first is that the conditional average turnover at the end of a period is directly related to the differences in ϑ across states. That is, the greater wealth transfer there is from state to state the more trading there is and this depends on the level of heterogeneity as given by both Δ and h. We see, therefore, that the total amount of trading in a given period is partly speculative and partly hedging and that:

(i) the amount of speculative trading is increasing in the level of heterogeneity, that is increasing in both the differences in beliefs Δ and the wealth dispersion h,

(ii) the amount of speculative turnover is decreasing in the risk-aversion $\gamma(\omega)$ and

(iii) the amount of hedging turnover is increasing in the risk-aversion $\gamma(\omega)$ and decreasing in the wealth dispersion.

The above three results will determine the relation between the amount of trading with prices, that is the conditional return volatility and the level of prices.

The second point is that the amount of turnover at the end of the period depends on the realization of ϵ, that is the amount of trading is state dependent. With this result we see in what way the structure of this economy is able to create the conditions for an asymmetric behavior in turnover. From a modeling point of view the element responsible for creating this condition is the introduction of the intermediate period, that is the separation of the trading due to new information and the trading due to the resolution of the period uncertainty. From an economic point of view it implies that this condition could be created if agents switch between speculative and hedging positions rather than always maintaining a balance between the two. For example this implies that whenever new information arrives in the economy then agents trade on that information and when the uncertainty for which that information was revealed is resolved then agents revert back to their hedging positions. Another way of putting it is to say that there are waves of speculative activity that is related to the arrival of information.

5.3 The Dynamic Relation Between Prices and Trading

After having analyzed how the different factors in this model economy determine prices and turnover it remains to see how prices and turnover are dynamically related, at first in the absence of time-variation in disagreement. The two stylized facts that are the focus of this paper are (i) the positive relation between volume and return volatility and (ii) the positive correlation between the stock return and volume.

Starting with the second empirical fact we have seen that the model is able to generate more trading within a period when the high endowment growth state is realized. We have also seen that prices are decreasing in the habit-endowment ratio ω which is driven by the negative endowment growth. Therefore, stock returns are positively correlated with the endowment growth and hence also with the stock turnover and in this way the model is able to generate the empirically observed pattern. The argument however can be made more general and for the argument to be true it requires two elements. The first is that occasionally investors receive economy-related information that they interpret differently and because of that they deviate from their hedging positions and then revert back to their hedging positions once the uncertainty for which they received the information has been resolved. Secondly, in doing so investors for whatever reason, for example to reduce costs related to trading, exchange such assets that the relatively optimistic investors decrease their holdings of the market portfolio and buy assets that pay well if the market goes up, for example call options. As a result, more trading occurs if the market indeed goes up in which case the previously optimistic agents increase their wealth and their holdings of the market portfolio. Note that in this explanation the positive correlation between market returns and the consumption growth is not required, since the call option is a contingent asset that depends on the market returns, whereas in the model it is required since we have contingent claims that depend on the realization of the endowment growth.

As for the volume-volatility relation the model has several factors that affect this relation in one way or another. The first factor is that in this model there are two types of trading which are the speculative due to the opinion differences and trading due to varying hedging demands coming from the habit preferences and the heterogeneity in wealth. The first thing to note is that the two types of turnover work in opposite directions, that is when the amount of the one increases the amount of the other typically decreases. In particular, speculative trading increases when dispersion in wealth (and the difference in beliefs) increases, that is when $h\Delta$ increases, whereas trading due to hedging increases when wealth heterogeneity increases, that is when h decreases. Also speculative turnover decreases with ω because risk-aversion decreases whereas trading due to hedging increases with ω because risk-aversion and in particular heterogeneity in risk-aversion increases. The overall effect depends, therefore, on the relative magnitude of these two trading activities and the effect of h, Δ and ω on volatility.

It has been seen that the habit-endowment ratio is the main asset pricing factor in determining the level of the market, return volatility, the equity premium and the risk-free rate. Effectively, the stock return volatility, the interest rate and the equity premium are counter-cyclical as they go up with ω and the stock price-dividend ratio is pro-cyclical.[36] Hence, speculative trading moves in the opposite direction to volatility whereas trading due to hedging demands in the same direction as volatility.

Belief heterogeneity, as given by $h\Delta$, is the second factor that affects return volatility. First, it needs to be noted that Δ has a smaller effect on the stock price and hence on its return volatility also due to its low persistence. The effect of h on the other hand is significant but two-fold, one that increases volatility and one that decreases it. Higher dispersion in wealth (lower h) also implies bigger wealth changes an effect that implies higher volatility. The second effect is through the persistence in h which is time-varying and decreasing in h. When there is high wealth dispersion, h has lower persistence which means that any change in h affects less the stock price which in turn implies lower return volatility. The overall effect depends on the model parameters; for the calibrated model return volatility is in fact decreasing in h but as it will be shown this relation reverses with a high ρ. A high value for ρ means that the average level of belief heterogeneity is high. This in turn means that the persistence of h decreases as well as its variation whereas its conditional volatility becomes more volatile and more important in affecting prices.

We will show later that for the calibrated model the overall relation between volume and volatility is at best weak. We will also show that even when we introduce the stochastic disagreement the model is still not able to generate the positive volume-volatility relation because of the effect of the time-varying risk-aversion, unless we choose a model configuration that is not able to fit prices. The model will only be able to generate both return-volume relations and fit prices well when we assume in addition that disagreement is positively correlated with risk-aversion.

5.4 Other Asset Structures

It has become obvious that the asset-structure has an important effect on the dynamic behavior of turnover especially in this model with the introduction of the intermediate state. For example a simple change in the state ϵ^* will result in a negative correlation between stock return and turnover. If this is an important factor behind the stylized fact it means that, for whatever reason, the relatively optimistic investors are the ones who decrease their holdings of the market portfolio by investing for example in call options rather than the relatively pessimistic investors either shorting the market or taking speculative positions in put options. Such an explanation sounds quite plausible if there are increased transaction costs in betting against the market. It is important to note that this explanation does not hint on market incompleteness but that the markets are completed with assets that pay-off when the stock return is high.

Typically asset pricing models assume asset structures that include a risk-free asset instead of derivatives on the stock or other contingent claims. Such an assumption in this model would

[36] In the model economy the economic cycle is determined by the habit-endowment ratio where high values for ω mean "bad times" and low values mean "good times".

produce no correlation between asset returns and turnover since the optimal holding of the stock would be given by

$$\theta(z,s) = \frac{W(L(z,s,+1)) - W(L(z,s,-1))}{P(L(z,s,+1)) - P(L(z,s,-1))}$$

in the intermediate periods. As a result the resulting turnover at the end of every period would typically not depend on the endowment growth and therefore largely uncorrelated with the stock return. This is an empirical question but it is likely that agents in their speculative trading do not switch between bonds and stocks since the trading volume on bonds is much smaller than the trading volume in either stocks or derivatives.

Even though not shown explicitly in this paper in general the positive relation between belief heterogeneity and trading volume is robust to the selection of the asset structure. The effect of belief heterogeneity, however, on the return volatility is not clear especially when volatility is mainly driven by other factors, as for example in this model by the habit-endowment ratio. Hence, it is not straight forward whether belief heterogeneity is able to produce a positive relation between volume and volatility. It will be shown that this relation can be produced without adversely affecting other price or volume characteristics of the model by introducing time-variability in disagreement.

6 Model Calibration

In order to analyze quantitatively the various effects that we presented and analyzed in the previous section, we calibrate the model to fit certain price and volume statistics. Both quarterly and annual data of turnover, market prices and consumption have been used. We obtained annual data on turnover from 1901 to 2003 from the NYSE Factbook. The annual data over the same period on per-capita consumption, market returns, the one-year risk-free rate and the market price-dividend ratio are from Robert Shiller's website. Our quarterly market data of turnover, market returns and market price-dividend ratio are from CRSP spanning the period from the first quarter of 1927 to the last quarter of 2007. The turnover is the total turnover from the entire stock universe of CRSP and the market price-dividend ratio was obtained from the CRSP value-weighted returns with and without dividends. The 3-month risk free rate was obtained for the same period from the Federal Reserve Economic Data of the St. Louis Fed. Quarterly per-capita real consumption data comes from the NIPA tables that are available only from the first quarter of 1947. Throughout the entire data period the turnover data are highly non-stationary due to the fact that they exhibit time-varying trend. For this reason we detrended the turnover series by taking first differences.

No specific calibration procedure was followed in the sense of fitting exactly certain statistics due to the high computational demand of solving and simulating one instance of the model. We aimed to fit closely a number of quantities like the annual mean, volatility and autocorrelation of consumption growth, the average risk-free rate and the average excess return on the market, the mean, volatility and autocorrelation of the market price-dividend ratio, the volatility of the risk free rate and the volatility of the stock return, the average Sharpe-ratio, the correlation between the stock return and turnover as well as certain statistics related to the volume-volatility relation. Table 1 shows the calibrated parameters in which the first row refers to the constant ρ model. All tables and figures appear in Appendix C.

The data and the calibrated model statistics are shown in Table 2. The last two columns of the table refer to the calibrated model with stochastic disagreement and will be discussed later. We obtained the model statistics by running 1000 simulations with 500 annual periods each. The column "Avg." shows the simulation average and "Std." the simulation standard deviation of each statistic.[37] Mean, volatility and first-lag autocorrelation are denoted with μ, σ and ϱ_1 respectively, r^f denotes the risk-free rate, r^m the stock return and pd the log market price-dividend ratio. Δx denotes the one-period change in a variable x and T denotes turnover. In order to analyze the calibrated model we plotted several equilibrium quantities in Figures 1 to 6 for different combinations of the state variables. In particular, we plotted the quantities against ω, α that determines $h(z) = \alpha(1-\alpha)$, $\pi = \pi^1 = \pi^2$ that determines $\mu(z)$ and combinations of π^1 and π^2 that determine $\Delta(z) = \pi^1 - \pi^2$.[38]

The model overall is able to produce a very good fit to the data excluding the statistics related to the volume-volatility relation. The consumption growth statistics are well fitted except the correlation with the stock return which is close to perfect in the model. The model risk-free rate exhibits higher volatility than that of the quarterly data but lower than the annual data volatility whereas the mean is fitted closer to the quarterly value. The annual data risk-free rate exhibits higher mean and volatility because of the fact that it is an annual interest rate instead of a three month interest rate and the data goes back to 1903 which was a period of exceptionally volatile interest rates. Looking at Figure 2 we observe that for the calibrated model the risk-free rate is mainly determined by ω and to a lesser extent by $\mu(z)$ due to the fact that ϕ_π was calibrated at a very low level, 0.056 to fit the consumption growth autocorrelation of 0.052. Further, the risk-free rate is hardly affected by belief heterogeneity either due to h or Δ.

The model statistics of the log market price-dividend ratio, that is the mean, volatility and persistence are very well fitted. Its main determining factor, as seen in Figure 1, is the habit-endowment ratio through its persistence and the way it determines the risk-free rate and the equity premium as seen in Figure 3. The mean and volatility of the stock returns and the market equity premium are fitted equally well. The persistence parameter ϕ_x lends its persistence to the price-dividend ratio and makes it quite volatile. The subjective discount factor δ was chosen to fit the level of the price-dividend ratio whereas the curvature parameter γ and the habit parameter β, that determine the risk-aversion, were chosen to fit the equity premium. Figure 1 also shows that belief heterogeneity, in particular h because of its persistence, affects negatively the stock price whereas Δ and $\mu(z)$ have very small effects. The same holds for the equity premium shown in Figure 3 but this time h has a noticeable effect because it decreases the correlation of the stock return with consumption growth.

[37] Standard errors can be obtained by dividing the standard deviation with $\sqrt{1000}$.

[38] Note that it is enough to plot the figures for values of α between 0.5 and 1 since the model is symmetrical in α. Wealth dispersion as measured by h increases when α decreases from 1 to 0.5 whereas wealth heterogeneity increases in the opposite direction.

Coming now to the statistics related to turnover, the model through choosing the parameter ρ is able to fit well the correlation of turnover with the stock returns, with the changes in the price-dividend ratio and with the excess stock returns. Also the model correlation of turnover with the risk-free rate is close to zero just like in the quarterly data whereas in the annual data it is negative. Another statistic that is not shown in Table 2 is the correlation of the price-dividend ratio with turnover which is also close to zero in both the annual data, 0.06, and the quarterly data, −0.03. This is not surprising since the turnover series was detrended through first differencing and hence it appears that no significant variation of its time-varying mean remains in the detrended series. Let the turnover series be decomposed into its time varying mean and its innovations with time-varying volatility:

$$T_{t,t+1} = \mathbb{E}_t \left(T_{t,t+1}\right) + \sqrt{\mathbb{V}_t \left(T_{t,t+1}\right)} \cdot \frac{T_{t,t+1} - \mathbb{E}_t \left(T_{t,t+1}\right)}{\sqrt{\mathbb{V}_t \left(T_{t,t+1}\right)}},$$

where $\mathbb{V}$ denotes variance. The price-dividend ratio and the risk-free rate being $\mathcal{F}_t-$measurable can only be related to the conditional expectation and the conditional variance of turnover and hence uncorrelated to the detrended series.

Since the turnover series was detrended there is no direct way of observing how the conditional mean of turnover relates to the level of prices. One indirect way to do it is to assume that the conditional volatility of turnover is proportional to its conditional mean and then see how the absolute turnover series is related to the log price-dividend ratio. The data show a strong negative correlation of −0.38 in the quarterly data and −0.45 in the annual data. This negative correlation is not surprising since return volatility is known to move in the opposite direction to prices while it is positively correlated to volume. The model on the other hand exhibits no correlation between turnover and the level of prices. The qualitative analysis of the model showed that there are two types of trading, speculative and hedging, and that there are two factors that affect this relation, ω and h. As the habit-endowment ratio is concerned speculative turnover is positively correlated and hedging turnover is negatively correlated with the price-dividend ratio. This is confirmed by Figure 5 where we see that when $\alpha = 0.5$ and therefore trading due to hedging is zero the expected one-period turnover decreases with ω. In the case where α is close to 1 in which case most of the trading is due to hedging the expected turnover is increasing in ω. The second factor h generates the opposite relations between the level of the market and the two types of trading. When ω is high in which case trading due to hedging is small the amount of trading increases with h and therefore speculative trading is negatively correlated with the price-dividend ratio. Trading due to hedging on the other hand decreases with h as we observe when ω is high and speculative trading is low. Judging from the overall zero correlation between turnover and the price-dividend ratio all these effects appear to cancel each other out.

For the same reason the model is unable to produce an unconditional relation between volume and volatility. But first lets review the statistical evidence with respect to the volume-volatility relation. In order to uncover such a relationship we consider a number of different statistical relations. First, the absolute stock returns and the absolute excess stock returns are both positively correlated with turnover. However, this is due to the high unconditional means of these returns and the positive correlation between these returns and turnover. For this reason we also look at the correlations between the absolute price-dividend ratio changes and the absolute demeaned returns with turnover. These correlations are positive in the quarterly data and close to zero in the annual data. Even though these pricing series are better in measuring pricing volatility the lack of strong evidence is probably due to the turnover measure. If the variation in turnover due to its positive correlation with price changes is a big part of the overall variation in trading then the correlation between absolute price changes and volume is not a good measure of the volume-volatility relation. For example, consider a case where the return volatility is high and the next-period return turns out to be low. Then even if there is a positive relation between volume and volatility the resulting volume would be low due to the low return. Therefore, it would be better to look at the conditional mean of turnover instead but for the reason that the turnover series is detrended it cannot be observed. Following the same reasoning as before an indirect evidence can be obtained by looking at the correlation between the absolute turnover with the pricing measures of volatility. In the quarterly data the correlations between $|\Delta pd|$ and $|r^m - \mu(r^m)|$ with $|T|$ are 0.40 and 0.37 respectively. In the annual data these correlations are 0.17 and 0.14 respectively. This statistical evidence is in line with the stylized fact of the positive relation between return volatility and turnover.

We have argued earlier that a positive volume-volatility relation requires a number of conditions in a model with realistic prices one of which is time-variation in disagreement as we will show later. We see here that the model with constant disagreement is unable to produce this effect as seen by the correlation of the stock turnover with either the absolute stock price-dividend ratio changes or the absolute demeaned stock returns. The model correlations of the absolute stock returns and the absolute excess stock returns with turnover are highly positive but as discussed earlier these are not good measures of the volume-volatility relation in the light of the positive correlation between returns and volume. The inability of the model with constant disagreement to generate this relationship can also be seen by looking at Figures 4 and 5. Clearly, the most important factor for the return volatility is the habit-endowment ratio and to a much lesser extent wealth dispersion. The conditional mean of the one-period turnover however does not behave uniformly in relation to ω since it is decreasing when h is high in which case trading is mostly speculative whereas it is increasing when h is low in which case trading is mostly for hedging purposes.

With the qualitative analysis we showed that in the model with constant disagreement the only way of generating a positive volume-volatility relation is if heterogeneity of beliefs as given by $h\Delta$ becomes an important factor for return volatility, return volatility becomes increasing in the belief heterogeneity and speculative turnover becomes much greater than hedging turnover. All these conditions could be met if the level of belief heterogeneity increased by increasing parameter ρ but with a number of other adverse effects. Figures 7 to 9 show how certain price characteristics and the price-volume relations are affected by varying parameter ρ from 0.05 to 0.45 and keeping the rest of the parameters the same. The plots were generated by solving each instant of the model and running 200 simulations with 500 periods each. The continuous lines show the simulation averages of the corresponding statistics and the dotted lines show the one standard error bounds.

Figure 9 verifies the conclusions from the qualitative analysis in that a strong volume-volatility relation can indeed be produced with a high value for ρ. However this has an adverse effect on the return-volatility relation, as shown in Figure 8, and on key pricing statistics. The stock-price and the risk-free rate volatilities increase significantly whereas the equity premium decreases drastically from close to 6% down to about 3%. When the heterogeneity of belief increases the risk-free rate becomes more volatile which in turn causes a higher volatility in the stock price. At the same time the correlation of the stock returns with the endowment growth decreases, since the variation of the stock price due to changes in h and Δ is generally unrelated to $y' - y$, which decreases the equity premium and the return-volume correlation. The latter effect can be remedied by substituting contingent claims on $y' - y$ with stock derivatives but the former effect cannot.

Lastly, the model independently of whether disagreement is time-varying is able to fit the positive autocorrelation of trading volume as measured by the first lag autocorrelation of the absolute turnover series. The logic behind looking at the absolute series of turnover is once again the fact that the series is detrended and the possibility that the conditional volatility of turnover is positively correlated with its conditional mean. The model generates this positive autocorrelation through the persistence in the level of belief heterogeneity, mainly due to wealth dispersion.

7 Stochastic Disagreement

The disagreement $|\rho|$ determines how differently the beliefs of the two agents are affected when the two signals revealed are conflicting. In particular, it affects the conditional volatility of the difference in beliefs as given by Δ where the random shock is the difference in the two signals:

$$\Delta(z, s) = \phi_\pi \Delta(z) + (1 - \phi_\pi)\rho(s^1 - s^2).$$

If one accepts that agents disagree occasionally when interpreting common information it is natural to also expect that the amount of disagreement will be time-varying. This could be because the economic uncertainty varies, or because the amount or rate of information release is different across time, or even due to psychological factors that are related to the state of the economy. All these factors could be captured in this model by a time-varying disagreement.

7.1 Additional Model Assumptions

We assume that the disagreement varies over time according to the following autoregressive process:

$$\rho_{t+1} = \phi_\rho \rho_t + \frac{1 - \phi_\rho}{2} \epsilon^\rho_{t+1}, \tag{25}$$

where $\phi_\rho \in (0,1)$ and

$$\epsilon^\rho_{t+1}|\epsilon^y_{t+1} = [2 \cdot \mathbb{1}(\rho_t \geqslant 0) - 1] \cdot \begin{cases} +1, & \text{w. probability } \eta, \\ -1, & \text{o/w } 1-\eta, \end{cases} \tag{26}$$

The shock ϵ^ρ realizes every whole period, that is at the same time as ϵ^y, and agents hold homogeneous beliefs as to its conditional distribution. The specific formulation of the shock ϵ^ρ is made so that η controls the correlation between endowment growth and changes to disagreement, $|\rho_{t+1}| - |\rho_t|$. Therefore an increase in disagreement implies a positive value for ϵ^ρ when ρ_t is positive and negative when ρ_t is negative. In the case where $\eta = 0.5$ the correlation is zero and therefore the term $[2 \cdot \mathbb{1}(\rho_t \geqslant 0) - 1]$ is redundant.[39] The introduction of the parameter η is important for generating the positive correlation between volume and volatility through affecting their joint dynamics. Its importance will be shown with the analysis of the calibrated model.

The state of the economy is now described by $z = (\alpha, \pi^1, \pi^2, \omega, \rho)$ and the shocks in the second sub-period are $\epsilon = (\epsilon^y, \epsilon^\rho)$. Even though ϵ^ρ introduces a non-fundamental source of uncertainty agents still require two additional independent assets for the second sub-period so that the financial markets are dynamically complete. This is because disagreement will have an effect on both prices and wealth even though it does not have an effect on consumption. The consumption surplus allocation remains the same which is shown in Lemma 5. The reason is that the consumption surplus processes depend on beliefs and the source of uncertainty for which agents hold heterogeneous beliefs and therefore unaffected by the introduction of ϵ^ρ.

The two additional assets in the second subperiod will be two contingent claims, one that has positive payoff in state $\epsilon = (+1, -1)$ and the other with positive payoff in state $\epsilon = (-1, -1)$. The particular asset structure is not important for the results obtained as long as $\{\epsilon^y = -1\} \in \epsilon^*$ as explained in Section 5.

7.2 Price and Volume Implications

A high level for disagreement implies that any possible disagreement when new information arrives will have have great impact on the difference in beliefs. We already saw that interest rates are affected by the heterogeneity in beliefs depending on the curvature parameter γ. The approximate expression of the additional element that we derived to show the impact of belief heterogeneity on the one-period risk-free rate is the following:

$$2h \frac{\sigma^2}{\sigma(z)^2} \frac{\gamma - 1}{\gamma} \left[(\phi_\pi \Delta(z))^2 + 2(1-\phi_\pi)^2 \rho^2 \right]. \tag{27}$$

The impact on the interest rate is very clear. In the relevant case where γ is above one higher disagreement implies higher expected heterogeneity in beliefs and hence higher risk-free rate and

[39] A simpler process could be specified that would make the term $[2 \cdot \mathbb{1}(\rho_t \geqslant 0) - 1]$ redundant if ρ was made to be always positive instead of also switching signs.

the opposite when γ is less than one.

Since interest rates are affected it means that the stock price will also be affected by the level of disagreement. Continuing with the relevant case where γ is greater than one, the stock price-dividend ratio will be decreasing in the disagreement and the magnitude of the effect will depend on the persistence ϕ_ρ. Higher persistence implies that for many periods ahead the disagreement is expected to be high increasing in this way the discounting of future cash-flows. Of course, on the other hand, higher ϕ_ρ implies smaller changes to the disagreement and therefore lower volatility due to changes in $|\rho|$. Another important effect of ϕ_ρ is on the unconditional average of the disagreement. The unconditional volatility of ρ is given by $\frac{1}{2}\sqrt{\frac{1-\phi_\rho}{1+\phi_\rho}}$ which is decreasing in ϕ_ρ. The unconditional average of the disagreement is related to the unconditional volatility of ρ and as such it is also decreasing in ϕ_ρ. As a result the average interest rate decreases whereas the equity premium and the correlation between the stock returns and the endowment growth increase with ϕ_ρ.

The return volatility is also affected by the variation in disagreement. The approximate expression (27) shows that the risk-free rate is proportional to ρ^2 and as a result the effect is increasing in $|\rho|$. Roughly, the reason is that when agents have different conditional means for the aggregate endowment growth it means that the same difference in beliefs applies equally to the high as well to the low state. The result is that since the interest rate is increasingly affected when $|\rho|$ increases it means that the stock price becomes increasingly sensitive to changes in the disagreement and hence its conditional volatility increases. As for the trading volume the effect is straight forward. In the approximate expressions derived in Section 5 the one-period turnover was shown to be proportional to the difference in beliefs $\Delta(z, s)$ which in turn is increasing in the disagreement. Simply put, higher disagreement implies bigger changes in beliefs, higher belief heterogeneity and bigger changes in wealth and therefore all of these imply higher expected speculative turnover. The variation in disagreement, therefore, creates a positive relation between volume and volatility but as we argued earlier and as we will show below it is still not enough to create a strong unconditional relation if we also need to explain prices.

7.3 Calibration

The parameters to be chosen for this calibration are the autocorrelation parameter ϕ_ρ and the parameter η that controls the correlation of changes to disagreement with aggregate endowment growth and hence the correlation between disagreement and the habit-endowment ratio. There is more than one combination of values for η and ϕ_ρ that can generate the two price-volume relations. For example η could be set to 0.5, that is no correlation between disagreement and ω, and ϕ_ρ could be set to 0.5 that would create high belief heterogeneity and high volatility in $|\rho|$. However, such a configuration generates a smaller correlation between return and volume, decreases the equity premium and increases both the risk-free rate and the stock price volatility.

In order to see the effect of ϕ_ρ we set η to 0.5 and vary the parameter ϕ_ρ from 0.5 to 0.95. Figure 16 shows that a decrease in ϕ_ρ increases the risk-free rate and the stock-price volatilities substantially while it also decreases by about 1% the equity premium. These effects are due to the fact that a decrease in ϕ_ρ increases the unconditional average of disagreement. As a result, belief heterogeneity increases and it becomes more volatile which increases the volatility of interest rates and decreases the correlation of the stock returns with endowment growth. This last effect has also a negative effect on the return-volume correlation as seen in Figure 17. Figure 18 on the other hand shows that a low value for ϕ_ρ does generate a significant correlation between volume and volatility even though not as strong as in the data.

The model configuration that offers the best fit to the data, both prices and volume, is one with positive correlation between disagreement and ω, and a high value for ϕ_ρ, that is a lower level of belief heterogeneity. The calibrated parameters are shown in the last row of Table 1 and the model statistics are shown in the last two columns of Table 2. The subjective discount factor β was also adjusted to fit the level of prices. The reason is that the calibrated model with the stochastic disagreement implies on average a lower level of disagreement than the calibrated model with constant ρ. Consequently, the risk-free rate with the same β parameter value would imply a lower average risk-free rate and a higher average price-dividend ratio.

The newly calibrated model fits very well the data and generates a strong relation between volume and volatility as indicated by the model statistics of $corr(|\Delta pd|, T)$ and $corr(|r^m - \mu(r^m)|, T)$ as well as the negative correlation between volume and the stock price-dividend ratio. In order to generate these strong relations the correlation between $|\rho|$ and ω was set to be positive with $\eta = 0.2$. A positive correlation between $|\rho|$ and ω means that when the economy enters into bad times and the volatility goes up the disagreement also goes up on average which increases speculative trading despite the increased risk-aversion. Despite the fact that the disagreement has a significant effect on return volatility and as such it creates a positive relation between volume and volatility, as shown by Figures 13 and 14, a zero correlation between $|\rho|$ and ω would give at best a weak volume-volatility relation. This is because ω creates a negative relation between volume and speculative turnover and cancels out the positive relation created by disagreement. Figure 21 shows how the volatility-volume relation changes when the parameter η varies from 0.5 to 0.1. It is clear that a model in which there is no correlation between ω and $|\rho|$, that is for $\eta = 0.5$, the volume-volatility relation is almost non-existent unconditionally with this level of disagreement whereas their correlation increases significantly as η decreases.

The parameter η also affects the return-volume relation as seen in Figure 20 but not as significantly as the other stylized fact. To understand the effect on the return-volume relation we first need to consider that the introduction of the stochastic disagreement decreased the correlation of the stock returns with the endowment growth and hence volume. However, by assuming a value for η other than 0.5 we can increase or decrease the correlation of changes to disagreement with the endowment growth and hence the correlation between stock returns and the endowment growth. If η is less than 0.5 it means that changes to the disagreement are negatively correlated with endowment growth and hence changes in the stock price due to changes in $|\rho|$ are positively correlated with $y' - y$ and hence volume.

Finally, the positive correlation assumed between ω and $|\rho|$ also affects the volatilities of the stock price and the risk-free rate. As seen in Figures 10 and 11 both ω and $|\rho|$ affect significantly both the price-dividend ratio and the risk-free rate and in both cases they work in the same direction. Hence, the positive correlation between ω and $|\rho|$ increases the overall volatilities. The effect, however, as observed by the unconditional model statistics is only marginal and does not have an adverse effect in the ability of the model to fit prices.

7.4 Discussion

The explanation proposed by this theoretical model is that the volume-volatility correlation can be positive and at the same time explain prices if risk-aversion and disagreement are positively correlated. Consequently, in economic downturns when the return volatility increases even though risk-aversion increases the amount of speculative turnover on average increases because disagreement is also likely to increase. In the absence of such a correlation or if disagreement is constant then risk-aversion generates a negative relation between volume and volatility which cancels the positive correlation generated either by the time-varying disagreement or the time-varying level of belief heterogeneity. If no correlation is assumed or if disagreement is constant the level of disagreement required to generate the volume volatility relation is too high with adverse pricing predictions. Interest rates become very volatile, stock prices as a response also become very volatile and their risk-premium decreases due to a decrease in their correlation with consumption.

As an assumption the positive correlation between risk-aversion and disagreement is plausible. The disagreement essentially determines how volatile the difference in beliefs can be within a given amount of time. Disagreement could be time-varying for many reasons, like the amount of information released or the impact of new information or even due to psychological factors that may affect the way agents interpret new information. It is possible that disagreement increases during bad times because investors become more attentive and more sensitive to information in the fear of making mistakes whereas in good times herding behavior may prevail. If this is so and if such behavior is strong enough to create a positive relation between volume and return volatility is an empirical question.

8 Conclusion

The theoretical model presented in this paper makes a step towards understanding in a unified framework the joint dynamic behavior of assets prices and the volume of trade that are empirically strongly related with each other. Such a theory is lacking in the asset pricing literature since dynamic asset pricing models are silent about volume whereas trading models do not attempt to explain the quantitative characteristics of prices. The paper apart from providing a model that is able to fit both prices and the price-volume relations also highlights two possibly important theoretical considerations in understanding financial markets. Incidentally, these two considerations are related to the two main price-volume relations.

The first theoretical consideration, which is related to the volume-volatility relation, is that the time-varying risk-aversion that helps explain asset prices and disagreement that helps explain volume both have very interesting and quantitatively important pricing and trading implications. In particular, we showed that when risk-aversion increases then trading due to hedging demands increases whereas speculative trading decreases. As a result, since speculative trading is greater in size, risk-aversion generates a counter-factual prediction about the relation between volume and return volatility. However, this effect cannot be taken in isolation since the primary reason behind trading is disagreement which generates a positive relation between volume and volatility. What is interesting, however, is that overall the unconditional relation between volume and volatility can be generated either by assuming a high level of disagreement or if we assume that disagreement is positively correlated with risk-aversion. While the former has adverse pricing predictions the latter allows us to also fit prices very well.

The second theoretical consideration is related to the correlation between stock returns and volume which empirically is positive. We showed that in a model where volume does not contain information such an asymmetric behavior can be produced depending on the informational structure as well as the asset structure. The information structure refers to the timing of information and the timing of economic shocks whereas the asset structure refers to the set of assets used for trading. The information structure matters because trading due to new information is different than trading due to shocks to the distribution of wealth. In our model agents receive economy related information, about which they occasionally disagree, and take speculative positions not at the same time but in between shocks to the aggregate endowment. We also assume that agents trade on a stock and assets that resemble call options and as a result the relatively optimistic agents decrease their holdings of the stock. When the period uncertainty is resolved, which is when the endowment shock is realized, volume is higher when the endowment is positively rather than negatively shocked. This is because a positive shock will increase the wealth of the relatively optimistic agents and also their holdings of the stock. If on the other hand there is a negative shock to the economy then the relatively pessimistic agents increase their wealth but the resulting volume is small because the relatively pessimistic agents had already increased their holdings of the stock when the information had arrived.

The paper has shown that both theoretical considerations are potentially important since the model presented is able to fit well prices and at the same time generate the two main price-volume relations. Consequently, our results support the view that if we are to understand the financial markets we need to empirically study the way information affects beliefs whether and how it generates disagreement and how disagreement varies over time.

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Appendix A Proofs

Proof of Remark 1. Use assumption (3) to compute the conditional probability of a signal given ϵ^π using

$$\mathbb{P}^i(s^j|\epsilon^\pi) = \sum_{s^k} \mathbb{P}^i(s^j, s^k|\epsilon^\pi), \qquad k \neq j$$

and in computing the absolute difference between the beliefs of the two agents for a given signal note that $|1 - 2 \cdot \mathbb{1}\{A\}| = 1$ for any event A. $\square$

Proof of Lemma 1. The probabilities $\mathbb{P}^i(\epsilon^\pi|s)$ are derived from the general formula:

$$\mathbb{P}^i(\epsilon^\pi|s) = \frac{\mathbb{P}^i(s|\epsilon^\pi)\mathbb{P}^i(\epsilon^\pi)}{\mathbb{P}^i(s)}$$

and note that $\mathbb{P}^i(\epsilon^\pi) = 1/2$ and $\mathbb{P}^i(s) = \sum_{\epsilon^\pi} \mathbb{P}^i(s|\epsilon_\pi) = 1/4$ for $i = 1, 2$. $\square$

Proof of Lemma 2. Let χ be an $\mathcal{F}_{t+1}-$measurable variable and note that

$$\mathbb{E}^i(\chi|\mathcal{F}_t) = \mathbb{E}\left[\mathbb{E}^i(\chi|\mathcal{F}_{t+0.5})\middle|\, \mathcal{F}_t\right]$$

where $\mathbb{P}(s_{t+0.5}|\mathcal{F}_t) = 1/4$ for all $s_{t+0.5}$. Also note that

$$\begin{aligned}\mathbb{E}^i(\chi|\mathcal{F}_{t+0.5}) = \mathbb{E}^i\left[\mathbb{E}(\chi|\pi_t)|\, \mathcal{F}_{t+0.5}\right] &= \mathbb{E}^i\left[\sum_{\epsilon_{t+1}} \chi(\epsilon_{t+1}, s_{t+0.5})\mathbb{P}(\epsilon_{t+1}|\pi_t)\middle|\, \mathcal{F}_{t+0.5}\right] \\ &= \sum_{\epsilon_{t+1}} \chi(\epsilon_{t+1}, s_{t+0.5})\mathbb{E}^i\left[\mathbb{P}(\epsilon_{t+1}|\pi_t)|\, \mathcal{F}_{t+0.5}\right]\end{aligned}$$

where

$$\mathbb{E}^i\left[\mathbb{P}(\epsilon_{t+1}|\pi_t)|\, \mathcal{F}_{t+0.5}\right] = \mathbb{P}^i(\epsilon^y_{t+1}|\mathcal{F}_{t+0.5})\mathbb{P}(\epsilon_{t+1}\backslash\epsilon^y_{t+1}|\epsilon^y_{t+1})$$

and $\mathbb{P}^i(\epsilon^y_{t+1}|\mathcal{F}_{t+0.5})$ is as given in the lemma. $\square$

Proof of Lemma 3. Market completeness and the homogeneity of beliefs about $t + 0.5$ implies that the marginal rate of substitution and hence growth in consumption surplus from time t to $t + 0.5$ is the same across agents:

$$\frac{2C^i_{t+0.5} - \beta X_t}{2C^i_t - \beta X_t} = p_t(s_{t+0.5}).$$

We then sum up across the two agents to obtain the following equilibrium condition:

$$\frac{2C^i_{t+0.5} - \beta X_t}{2C^i_t - \beta X_t} = \frac{C_{t+0.5} - \beta X_t}{C_t - \beta X_t},$$

where C denotes the total consumption of the two agents in a given subperiod and note that $C_{t+0.5} = b^1_t + b^2_t$. Therefore, consumptions at time $t + 0.5$ are the same across states and known at time t. Further, taking into consideration condition (12) we see that the stochastic discount factor is equal to one which implies that,

$$\widetilde{P}_t^j = \mathbb{E}_t\left[\widetilde{R}_{t+0.5}^j\right],$$

for all assets j and for all whole periods t. Further, $p_t(s_{t+0.5}) = \delta^{1/2\gamma}$ which implies that:

$$C_t^i - \beta\frac{X_t}{2} = \frac{C_{t+0.5}^i + C_t^i - \beta X_t}{1 + \delta^{1/2\gamma}}.$$

Finally from the fact that $C_{t+0.5}^i$ is known at time t and independent of the state $s_{t+0.5}$ and given the budget constraint at time $t + 0.5$ as given in the optimization problem $(\mathcal{O})$ it must be that

$$C_{t+0.5}^i = b_t^i.$$

□

Proof of Lemma 4. Let

$$M_{t,t+1}^i = \delta\left(\frac{2C_{t+1}^i - \beta X_{t+1}}{2C_t^i - \beta X_t}\right)^{-\gamma}$$

which from Lemma 3, the definition of α^i in (13) and the definition of ω in (14) can be written as

$$M_{t,t+1}^i = \left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma}\left(\frac{1 - 2\beta\exp(\omega_{t+1})}{1 - 2\beta\exp(\omega_t)}\right)^{-\gamma}\left(\frac{\alpha_{t+1}^i}{\alpha_t^i}\right)^{-\gamma}.$$

Now, note from the first order conditions (10) and (11) and that fact that $R_{t+0.5}^1 = P_{t+0.5}$ that $M_{t,t+1}^i$ is agent i's one period stochastic discount factor for the stock. Finally, note from result of Lemma 3 that $M_{t,t+1}^i$ is also the half-period stochastic discount factor but given the new information set $\mathcal{F}_{t+0.5}$. The exponent of the aggregate endowment growth $1 - \gamma$ is obtained by dividing both sides with Y_t to obtain the equilibrium condition for the price-dividend ratio. □

Proof of Lemma 5. Financial markets are dynamically complete and therefore all state prices are the same across agents which implies that

$$\mathbb{P}^1(s_{t+0.5}, \epsilon_{t+1}|\mathcal{F}_t)M_{t,t+1}^1 = \mathbb{P}^2(s_{t+0.5}, \epsilon_{t+1}|\mathcal{F}_t)M_{t,t+1}^2$$

where $\mathbb{P}^i(s_{t+0.5}, \epsilon_{t+1}^y|\mathcal{F}_t)$ is as given in Lemma 2 and $M_{t,t+1}^i$ as given at the proof of Lemma 4. The result is obtained by rearranging the above equation. □

Proof of Lemma 6. Given the result of Lemma 3 that $C_{t+0.5}^i = b_t^i$ and the budget constraint in period $t + 0.5$ and the fact that it is binding in equilibrium we have that

$$\bar{\theta}_{t+0.5}^i\mathbb{P}_{t+0.5} = \bar{\theta}_t^i R_{t+0.5} = \mathbb{E}_{t+0.5}^*\left[M_{t,t+1}^*\bar{\theta}_{t+0.5}^i R_{t+1}\right]$$

where

$$M^*_{t,t+1} = \xi(\epsilon^y|z,s)e^{-\gamma(y'-y)}\left(\frac{1-2\beta e^{\omega'}}{1-2\beta e^{\omega}}\right)^{-\gamma}$$

as shown in equation (17). We also have that $\bar{\theta}^i_t \mathbb{P}_t = \mathbb{E}_t\left[\bar{\theta}^i R_{t+0.5}\right]$ from the fact that $P^j_t = \mathbb{E}[R^j_{t+1}]$. Let $W_t = \bar{\theta}^i_{t-0.5}R_t$ and $W_{t+0.5} = \bar{\theta}^i_t R_{t+0.5}$, apply conditional expectations to the budget constraints and divide both sides with Y_t to obtain the result. □

Proof of Remark 2. The first property is an outcome of Hölder's inequality and it's reverse. The other two properties are obtained with partial differentiation and using the fact that $\pi^1 > \pi^2$. □

Appendix B Approximations

For a given state $z = (\alpha, \pi^1, \pi^2, \omega)$ let $\Delta(z) \coloneqq \pi^1 - \pi^2$ and $h(z) \coloneqq \alpha(1-\alpha)$. The law of motion of the state vector is denoted by $z' = L(z, s, \epsilon)$. $\Delta(z, s)$ represents the corresponding variable in the middle of a period when signal s is observed. The log-consumption growth is $y' - y$ and let $\mu(z)$ and $\sigma(z)$ denote the conditional mean and volatility of log-consumption growth.

The stochastic discount factor is given by

$$M(z,z') = \delta\left(\frac{e^{y'-y} - 2\beta e^{\phi_x\omega}}{1-2\beta e^{\omega}}\right)^{-\gamma}\xi(\epsilon^y|z,s).$$

The first component using Taylor expansion wrt. $y'-y$ around $-(1-\phi_x)\omega$ can be expressed as:

$$\left(\frac{e^{y'-y} - 2\beta e^{\phi_x\omega}}{1-2\beta e^{\omega}}\right)^{-\gamma} = e^{\gamma(1-\phi_x)\omega}\Bigg[1 - \gamma(\omega)(y'-y+(1-\phi_x)\omega) + \frac{\gamma(\omega)^2}{2}(y'-y+(1-\phi_x)\omega)^2\left(1+\frac{1}{\gamma}-\frac{1}{\gamma(\omega)}\right) + \mathcal{O}((y'-y)^3)\Bigg],$$

where $\gamma(\omega) \coloneqq \gamma/(1-2\beta e^{\omega})$ indicates the level of risk-aversion in state z. The second element is given by

$$\xi(\epsilon^y|z,s) = \frac{q(\epsilon^y|z,s)}{\mathbb{P}^*(\epsilon^y|z,s)}$$

where q can be expressed using Taylor expansion for each $(\mathbb{P}^i)^{1/\gamma}$ around $\mathbb{P}^* = \alpha\mathbb{P}^1 + (1-\alpha)\mathbb{P}^2$ to get

$$\xi(\epsilon^y|z,s) = \left[1 + h(z)\frac{1-\gamma}{2\gamma^2}\left(\frac{\Delta(z,s)}{\mathbb{P}^*(\epsilon^y|z,s)}\right)^2 + \mathcal{O}\left(\left(\frac{\Delta(z,s)}{\mathbb{P}^*(\epsilon^y|z,s)}\right)^3\right)\right]^{\gamma},$$

and then using again Taylor expansion wrt. $\Delta(z,s)$ around zero to get

$$\xi(\epsilon^y|z,s) = 1 + h(z)\frac{1-\gamma}{2\gamma}\left(\frac{\Delta(z,s)}{\mathbb{P}^*(\epsilon^y|z,s)}\right)^2 + \mathcal{O}\left(\Delta(z,s)^3\right).$$

Applying conditional expectations to both sides:

$$\mathbb{E}\left[\xi(\epsilon^y|z,s)|z\right] = 1 + 2h(z)\sigma^2\frac{1-\gamma}{\gamma}\mathbb{E}\left[\left.\frac{\Delta(z,s)^2}{\sigma(z,s)^2}\right| z\right] + \mathcal{O}\left(\Delta(z,s)^3\right).$$

From the expressions derived for the stochastic discount factor the continuously compounded risk-free rate can be approximated by:

$$\begin{aligned} r^f(z) \approx & -\log(\delta) + \gamma(\omega)\mu(z) + [\gamma(\omega)-\gamma](1-\phi_x)\omega - 2h(z)\sigma^2\frac{1-\gamma}{\gamma}\mathbb{E}\left[\left.\frac{\Delta(z,s)^2}{\sigma(z,s)^2}\right| z\right] \\ & - \frac{\gamma(\omega)^2}{2}\left[\sigma(z)^2 + (\mu(z)+(1-\phi_x)\omega)^2\right]\left(1+\frac{1}{\gamma}-\frac{1}{\gamma(\omega)}\right). \end{aligned} \tag{B1}$$

If the variation in the conditional volatility of endowment growth is neglected then the belief heterogeneity component in the approximate risk-free rate expression can be substituted with the following:

$$-2h(z)\frac{\sigma^2}{\sigma(z)^2}\frac{1-\gamma}{\gamma}\left[(\phi_\pi\Delta(z))^2 + 2(1-\phi_\pi)^2\rho^2\right],$$

since $\Delta(z,s) = \phi_\pi\Delta(z) + (1-\phi_\pi)\rho(s^1 - s^2)$.

The wealth share of the first agent in state z is denoted with $\vartheta(z) := \widetilde{W}(z)/(\widetilde{P}(z)+1)$ and θ denotes the equilibrium holding of the market security by the first agent which for state z is given by

$$\theta(z) = \frac{\widetilde{W}(z,s^*)}{\widetilde{P}(z,s^*)} = \vartheta(z) + \lambda(z,s^*) \tag{B2}$$

where s^* is the state for which there does not exist a contingent claim with positive payoff and

$$\lambda(z,s) = \frac{1}{\widetilde{P}(z,s^*)}\mathbb{E}\left\{\left[\vartheta(z') - \vartheta(z)\right]M(z,z')e^{y'-y}\left[\widetilde{P}(z')+1\right]\Big| z,s\right\}. \tag{B3}$$

Equivalently $\lambda(z)$ is defined as the above expectation but given state z and thus,

$$\vartheta(z) = c(z) + \lambda(z)\widetilde{P}(z), \tag{B4}$$

where $c(z)$ is the consumption share of the first agent in state z and is equal to $\alpha(1-2\beta e^\omega)+\beta e^\omega$. The quantity λ is non-zero to the extend that wealth share changes are correlated with discounted prices. In the case where $\beta = 0$ and $\gamma = 1$ the correlation is identically zero whereas for γ's other than one it becomes non-zero and in certain middle of the period states it becomes significant. When β is positive then this correlation increases because wealth share changes are correlated with the habit-endowment surplus ratio.

In state (z,s) the optimal stock holding for the first agent is given by $\theta(z,s) = \vartheta\left(L(z,s,\epsilon^*)\right)$ where ϵ^* is the state for which there does not exist a contingent claim with positive payoff. Hence the equilibrium volume of trade in the market security over a period is given by

$$T(z,s) = \left|\vartheta\left(L(z,s,\epsilon^*)\right) - \vartheta(z) - \frac{\lambda(z,s^*)}{\widetilde{P}(z,s^*)}\right|$$

and

$$T(z,s,\epsilon) = \left|\vartheta\left(L(z,s,\epsilon)\right) + \frac{\lambda\left(L(z,s,\epsilon),s^*\right)}{\widetilde{P}\left(L(z,s,\epsilon),s^*\right)} - \vartheta\left(L(z,s,\epsilon^*)\right)\right|.$$

The most important component of the volume of the first round of trade is the change in the wealth share of which the most important part is the change in the consumption share. Similarly, the volume of the second round of trade in the case where ϵ is different than ϵ^* is mostly given by the difference in the consumption share between the two states.

Finally, using Taylor expansion of the probabilities $(\mathbb{P}^i)^{1/\gamma}$ around $\mathbb{P}^*$ the law of motion of the consumption surplus share is expressed by

$$\alpha' = \alpha + h(z)\frac{\Delta(z,s)\epsilon^y}{\gamma\mathbb{P}^*(\epsilon^y|z,s)} f_1(z,s),$$

where

$$f_1(z,s) = \frac{1 + \dfrac{1-\gamma}{2\gamma}\dfrac{\Delta(z,s)\epsilon^y}{\mathbb{P}^*(\epsilon^y|z,s)}(1-2\alpha)}{1 + h(z)\dfrac{1-\gamma}{2\gamma^2}\dfrac{\Delta(z,s)^2}{\mathbb{P}^*(\epsilon^y|z,s)^2}} + \mathcal{O}(\Delta(z,s)^3), \tag{B5}$$

which is identically one when $\gamma = 1$ or when $\Delta(z,s) = 0$.

Appendix C Tables and Figures

Table 1: Calibrated model parameters.

	μ	σ	ϕ_π	ϕ_x	δ	β	γ	κ	ρ	ϕ_ρ	η
Constant ρ	2.01	3.37	0.056	0.90	0.995	0.425	3.50	0.10	0.15	-	-
Stochastic ρ	2.01	3.37	0.056	0.90	0.988	0.425	3.50	0.10	-	0.95	0.20

The two rows refer to the two corresponding model specifications considered. The calibration aimed to fit first the annual per-capita consumption growth statistics, the annualized 3-month risk-free rate, the market price-dividend ratio and the annual market returns and equity premium statistics as well as the statistics related to the relation between turnover and prices. The data and the calibrated model statistics are shown in Table 2. Annual data on per-capita consumption, market returns, the one-year risk-free rate and the market price-dividend ratio were obtained from Robert Shiller's website and cover the period from 1901 to 2003. Annual data on turnover were obtained from the NYSE Factbook for the same period. The quarterly market data on turnover, market returns and market price-dividend ratio were obtained from CRSP spanning the period from the first quarter of 1927 to the last quarter of 2007. The turnover is the total turnover from the entire stock universe of CRSP and the market price-dividend ratio was obtained from the CRSP value-weighted returns with and without dividends. The 3-month risk free rate was obtained for the same period from the Federal Reserve Economic Data of the St. Louis Fed. Quarterly per-capita real consumption data comes from the NIPA tables that are available only from the first quarter of 1947.

Table 2: Data and calibrated model statistics.

	Data		Constant ρ		Stochastic ρ	
	Quarterly	Annual	Avg.	Std.	Avg.	Std.
$\mu(\Delta y)$	1.89	2.01	2.01	(0.15)	2.01	(0.15)
$\sigma(\Delta y)$	1.33	3.37	3.37	(0.01)	3.37	(0.01)
$\varrho_1(\Delta y)$	0.27	0.05	0.05	(0.04)	0.05	(0.05)
$corr(\Delta y, r^m)$	0.27	0.59	0.95	(0.01)	0.94	(0.01)
$corr(\Delta y, r^m - r^f)$	0.23	0.60	0.98	(0.00)	0.98	(0.00)
$\mu(r^f)$	0.68	1.51	0.45	(0.75)	0.49	(0.74)
$\sigma(r^f)$	2.49	5.32	3.70	(0.56)	3.91	(0.60)
$\varrho_1(r^f)$	0.58	0.49	0.87	(0.03)	0.88	(0.03)
$corr(r^f, T)$	-0.03	-0.17	0.03	(0.08)	0.30	(0.06)
$\mu(pd)$	3.35	3.20	3.25	(0.10)	3.21	(0.08)
$\sigma(pd)$	0.44	0.42	0.42	(0.05)	0.42	(0.05)
$\varrho_1(pd)$	0.86	0.85	0.91	(0.02)	0.90	(0.02)
$corr(\Delta pd, T)$	0.42	0.40	0.45	(0.09)	0.46	(0.06)
$corr(\|\Delta pd\|, T)$	0.29	-0.10	-0.00	(0.15)	0.34	(0.09)
$corr(pd, \|T\|)$	-0.38	-0.45	-0.02	(0.15)	-0.29	(0.08)
$\mu(r^m)$	6.50	6.09	6.19	(0.31)	6.33	(0.27)
$\sigma(r^m)$	21.07	18.14	20.51	(0.76)	20.80	(0.70)
$\varrho_1(r^m)$	-0.05	0.06	0.02	(0.05)	0.02	(0.05)
$corr(r^m, T)$	0.42	0.52	0.46	(0.08)	0.47	(0.06)
$corr(\|r^m\|, T)$	0.30	0.23	0.33	(0.08)	0.48	(0.05)
$corr(\|r^m - \mu(r^m)\|, T)$	0.22	-0.01	-0.00	(0.15)	0.30	(0.09)
$\mu(r^m - r^f)$	5.81	4.58	5.73	(0.51)	5.84	(0.52)
$\sigma(r^m - r^f)$	21.25	18.18	19.82	(0.70)	20.05	(0.62)
$\varrho_1(r^m - r^f)$	-0.05	0.07	-0.01	(0.05)	-0.01	(0.05)
$\mu(r^m - r^f)/\sigma(r^m - r^f)$	0.27	0.25	0.29	(0.03)	0.29	(0.03)
$corr(r^m - r^f, T)$	0.42	0.56	0.47	(0.10)	0.42	(0.06)
$corr(\|r^m - r^f\|, T)$	0.30	0.19	0.39	(0.08)	0.51	(0.05)
$\varrho_1(\|T\|)$	0.31	0.24	0.32	(0.09)	0.36	(0.07)

*The model statistics are obtained from 1000 simulations of 500 annual periods each. Avg. and Std. are the average and the standard deviation respectively of each statistic from the simulations. Standard errors can be computed by dividing the Std. statistic by $\sqrt{1000}$. The first two columns show the annual (1901-2003) and quarterly (1927:1-2007:4) statistics respectively. For a description of the data see Table 1. The third and fourth columns refer to the constant ρ model specification and the last two columns to the stochastic ρ specification. The notation $\mu(x)$, $\sigma(x)$ and $\varrho_1(x)$ denote the sample mean, volatility and first-lag autocorrelation of a variable x and $corr(x_1, x_2)$ denotes the sample correlation between two variables x_1 and x_2.

Figure 1: Constant ρ model - Log market price-dividend ratio pd

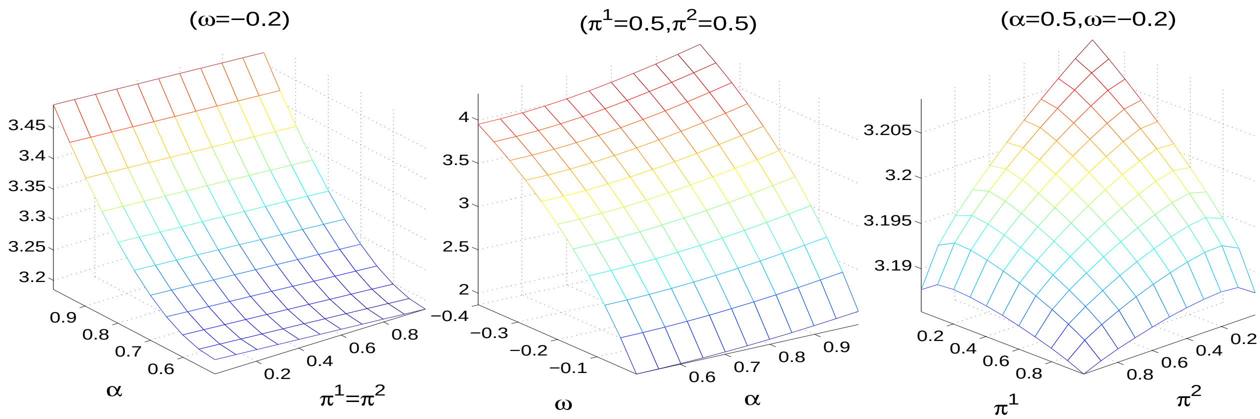

Figure 2: Constant ρ model - One-period risk free rate r^f

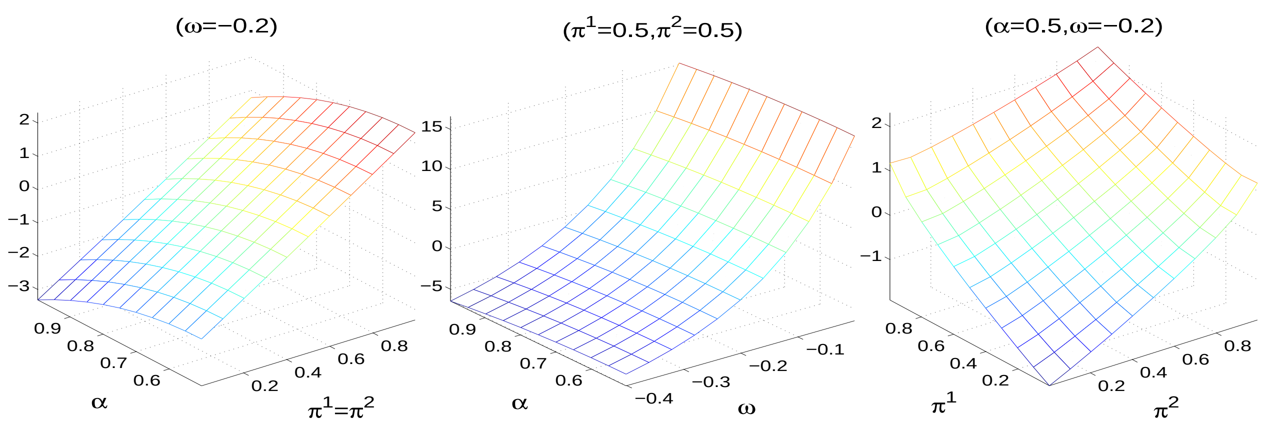

Figure 3: Constant ρ model - Market one-period expected excess return $\mathbb{E}(r^m - r^f)$

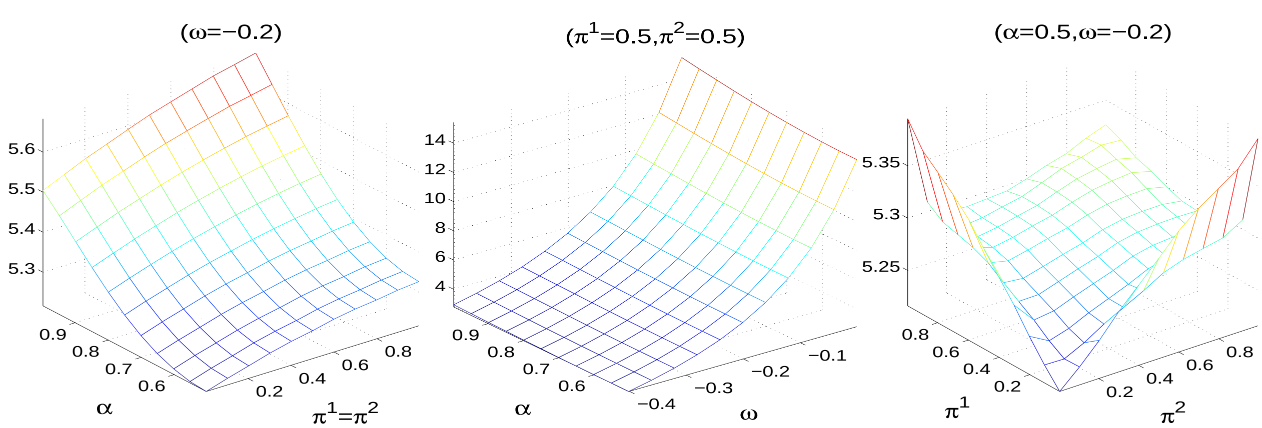

Figure 4: Constant ρ model - One-period market return volatility $\sigma(r^m)$

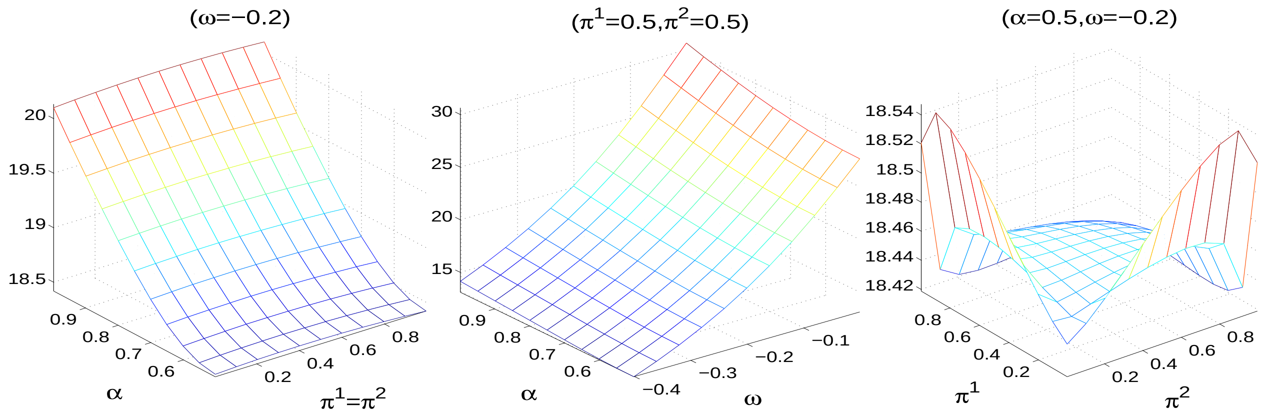

Figure 5: Constant ρ model - One-period market security expected turnover $\mathbb{E}(T)$

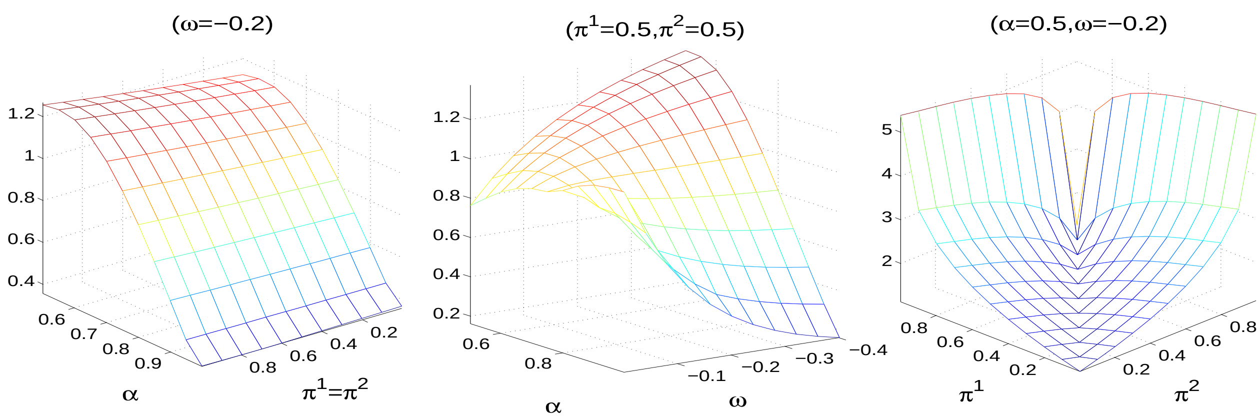

Figure 6: Constant ρ model - First agent's market security holding θ

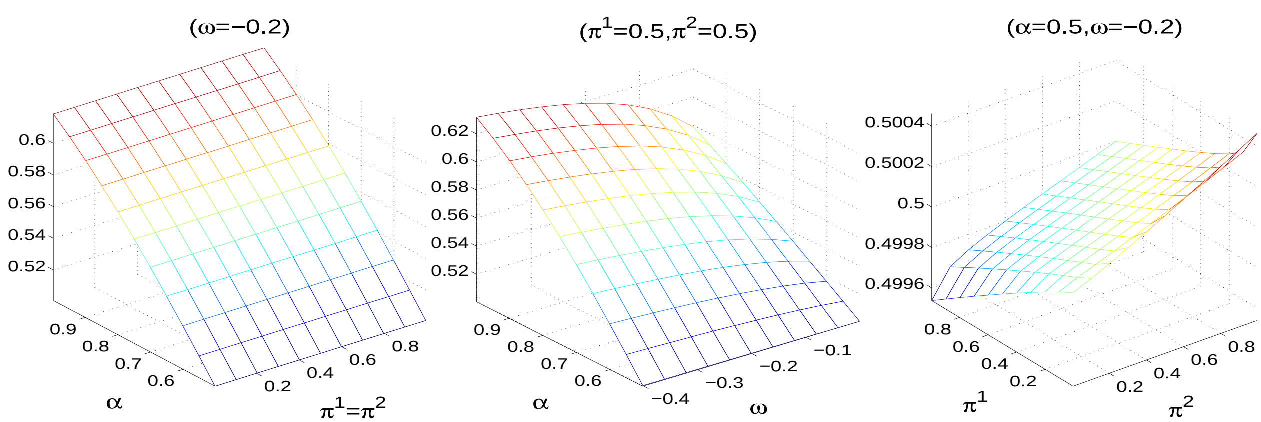

Figure 7: Constant ρ model - Impact of ρ on prices

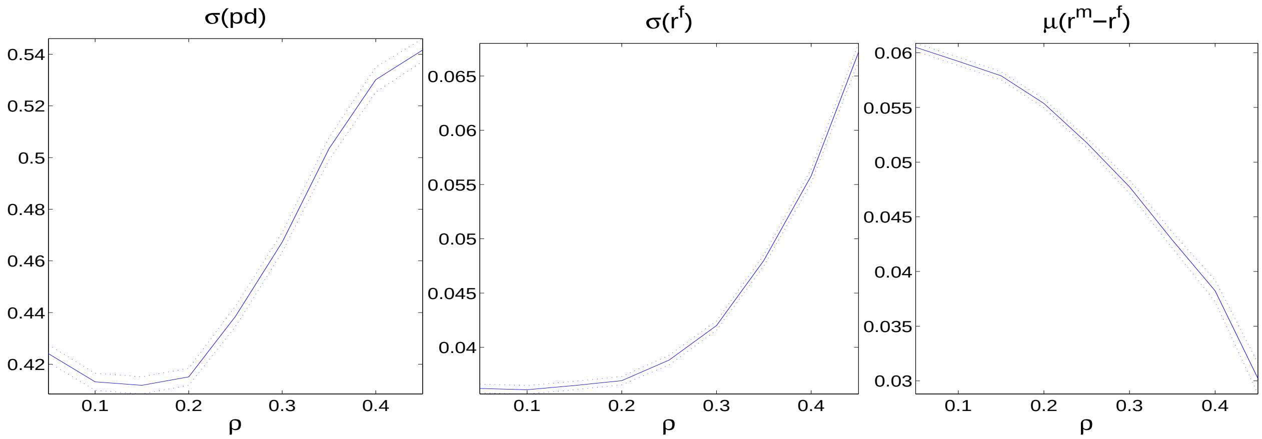

Figure 8: Constant ρ model - Impact of ρ on the return-volume relation

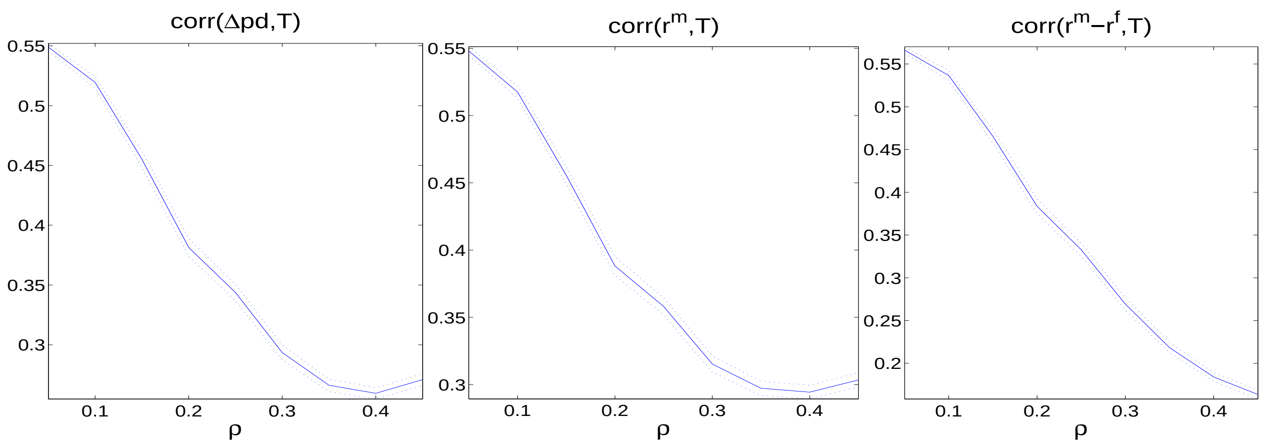

Figure 9: Constant ρ model - Impact of ρ on the volatility-volume relation

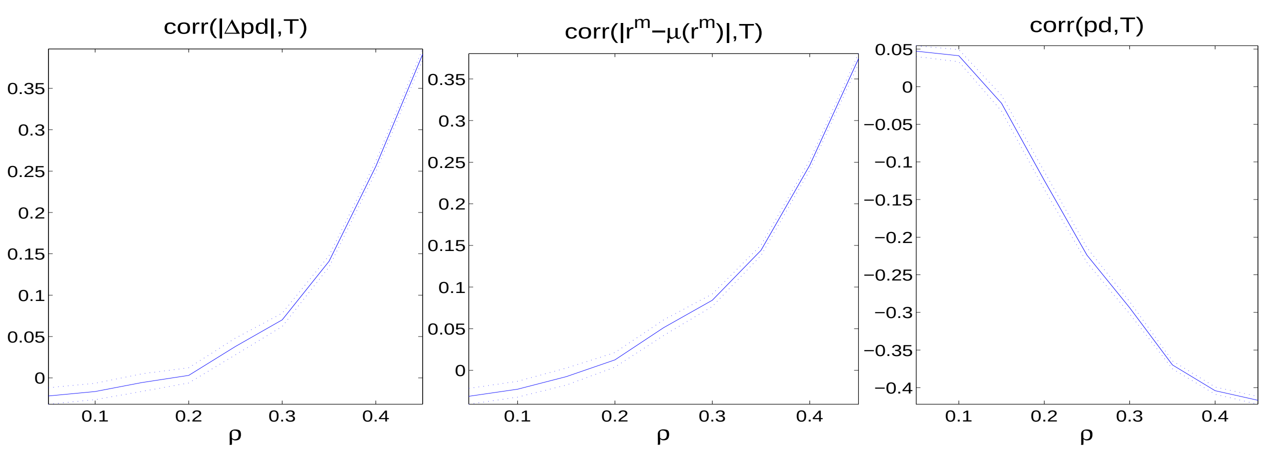

Figure 10: Stochastic ρ model - Log market price-dividend ratio pd

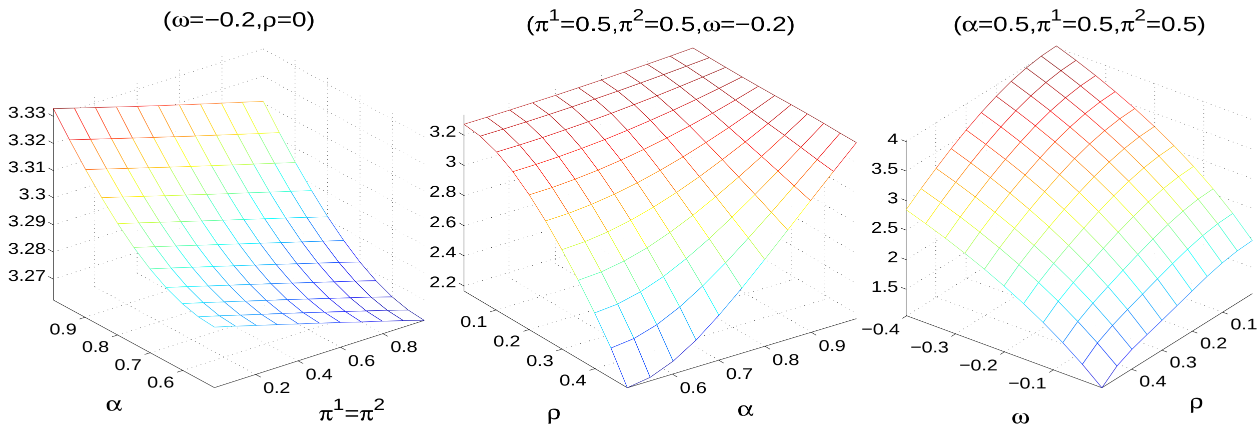

Figure 11: Stochastic ρ model - One-period risk free rate r^f

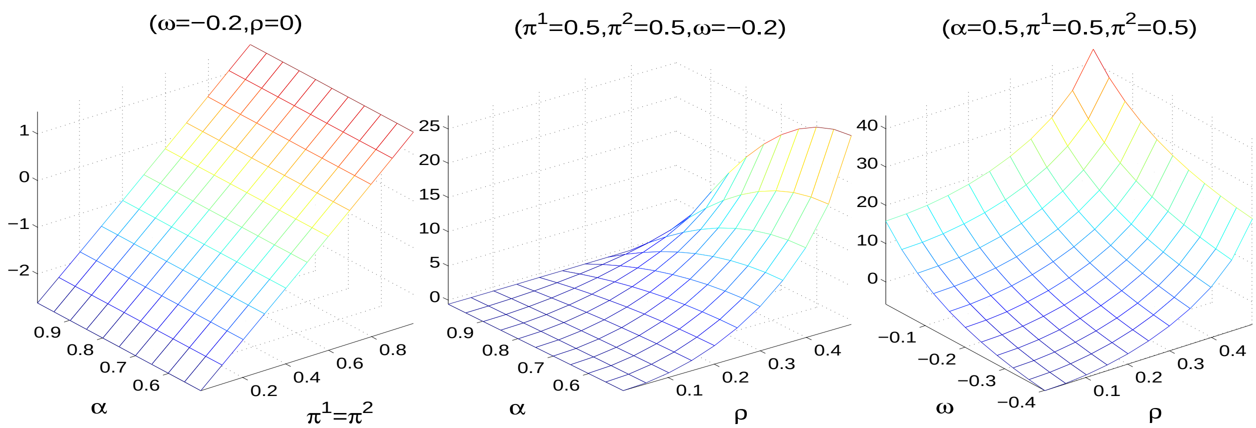

Figure 12: Stochastic ρ model - Market one-period expected excess return $\mathbb{E}(r^m - r^f)$

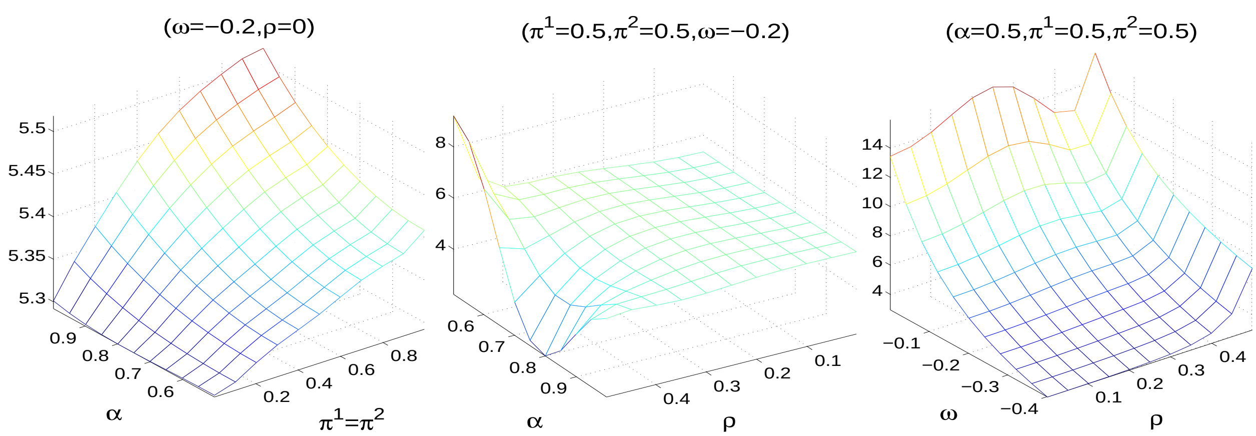

Figure 13: Stochastic ρ model - One-period market return volatility $\sigma(r^m)$

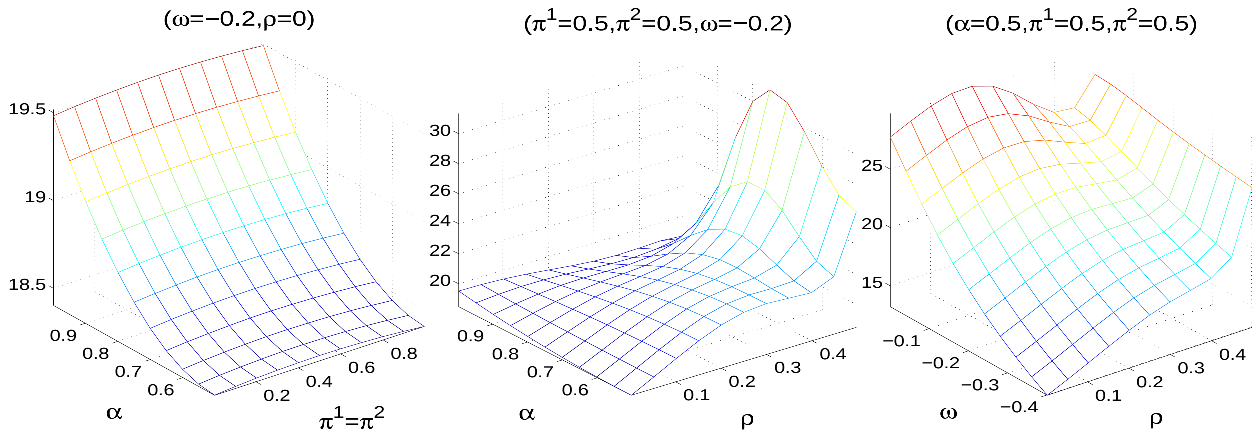

Figure 14: Stochastic ρ model - One-period market security expected turnover $\mathbb{E}(T)$

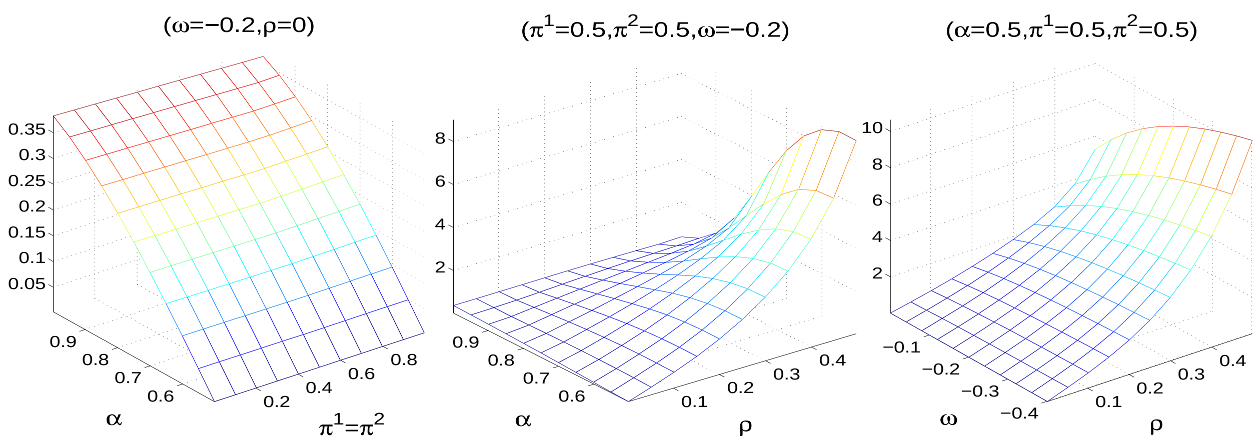

Figure 15: Stochastic ρ model - First agent's market security holding θ

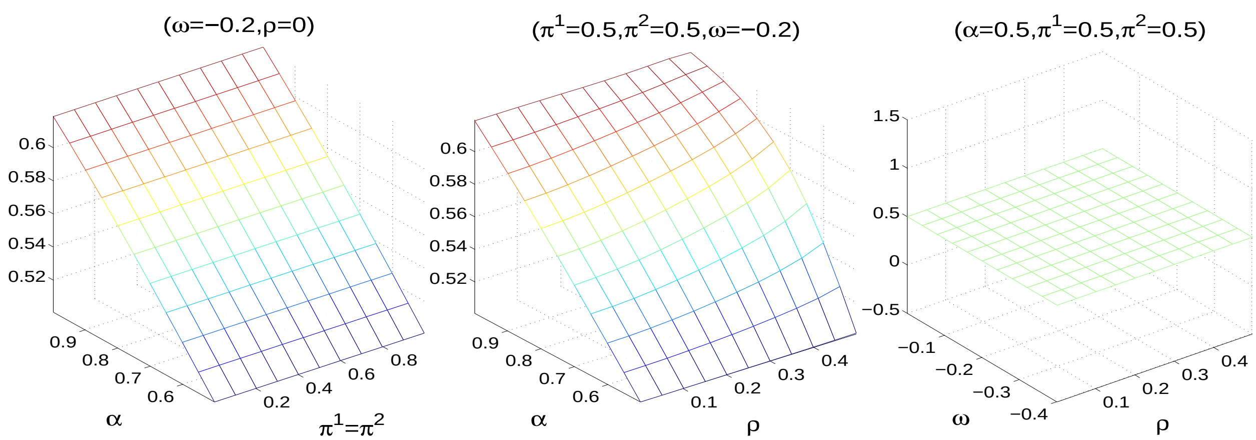

Figure 16: Stochastic ρ model - Impact of ϕ_ρ on prices ($\eta = 0.5$)

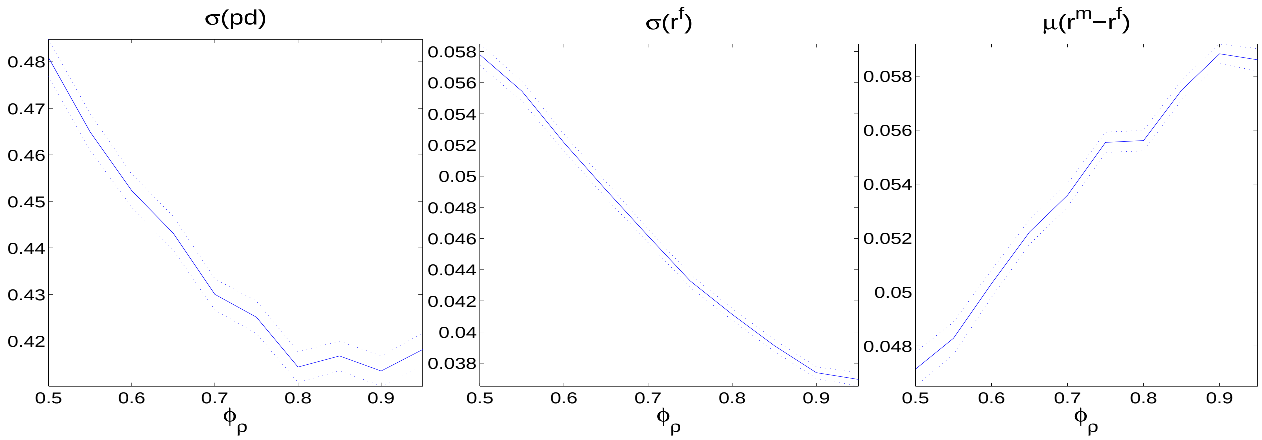

Figure 17: Stochastic ρ model - Impact of ϕ_ρ on the return-volume relation ($\eta = 0.5$)

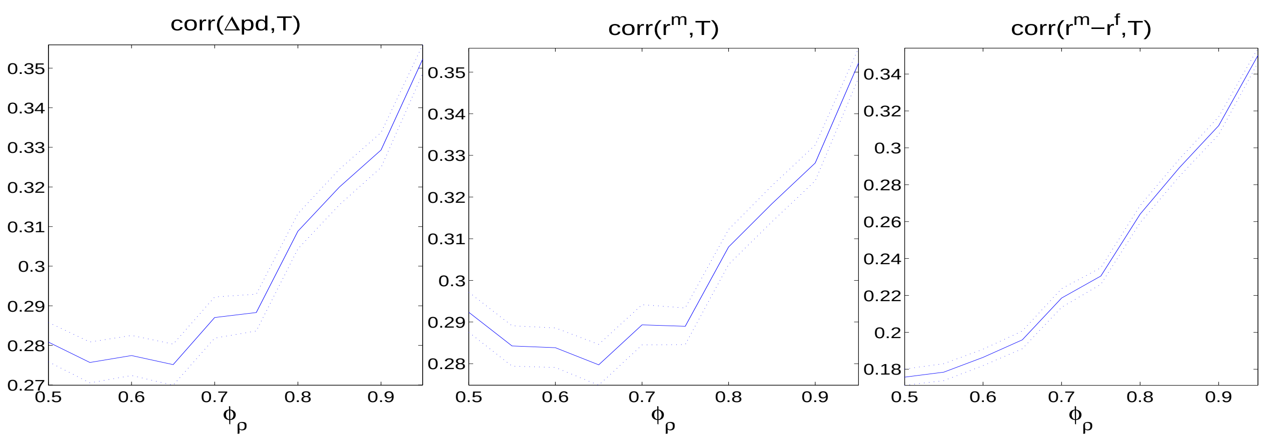

Figure 18: Stochastic ρ model - Impact of ϕ_ρ on the volatility-volume relation ($\eta = 0.5$)

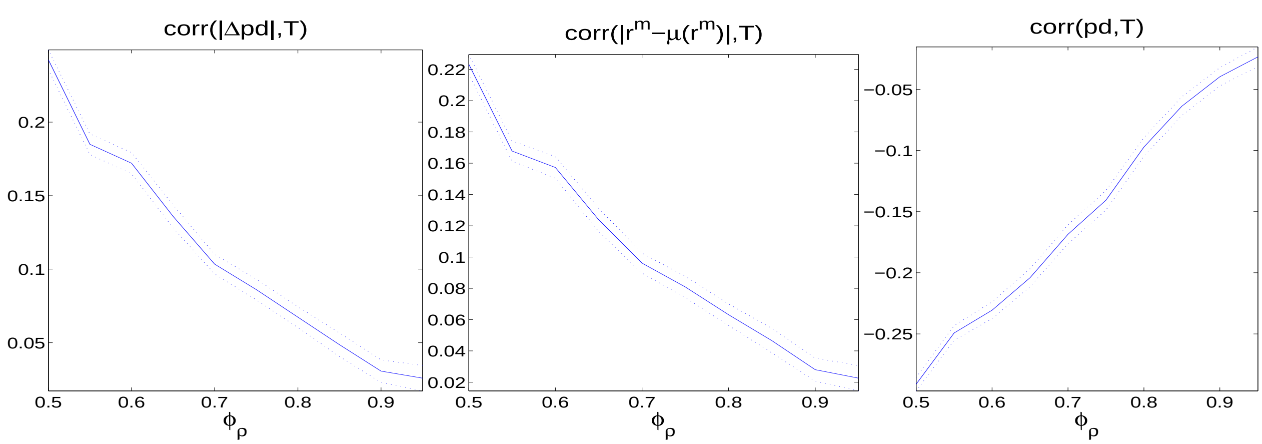

Figure 19: Stochastic ρ model - Impact of η on prices

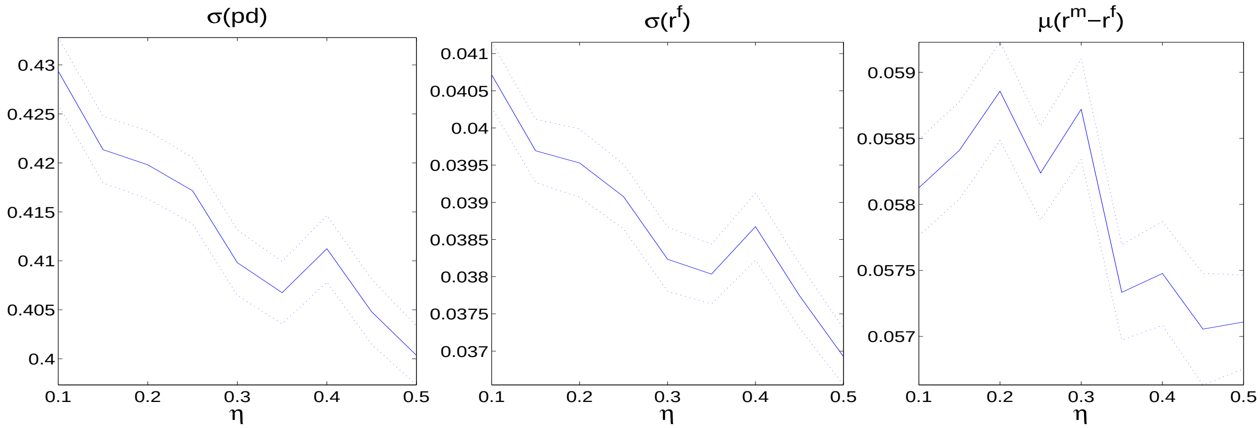

Figure 20: Stochastic ρ model - Impact of η on the return-volume relation

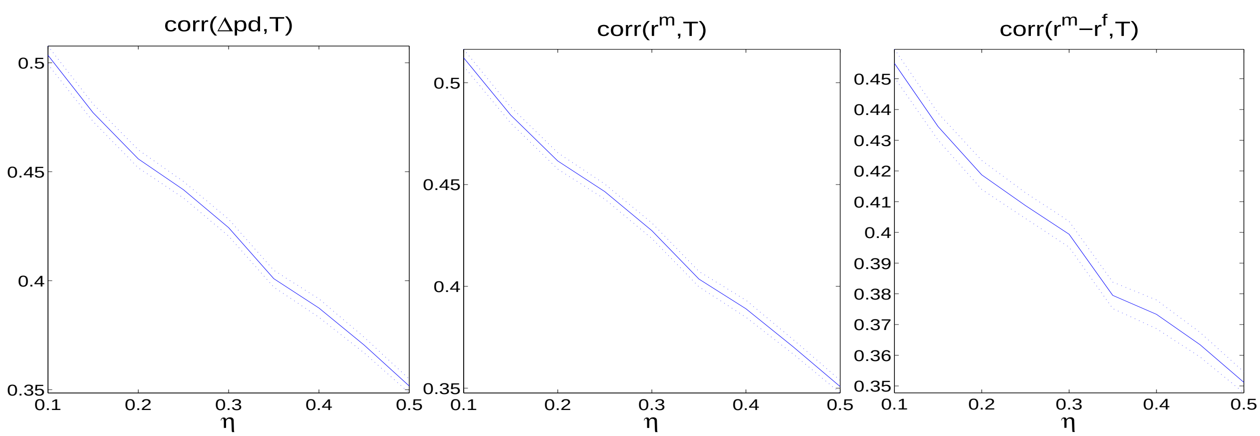

Figure 21: Stochastic ρ model - Impact of η on the volatility-volume relation