A Network-Based Analysis of Over-the-Counter Markets

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Abstract

I study how intermediation in over-the-counter markets affects the efficiency of resource allocation. I model an over-the-counter market as a trading network in which bilateral prices and agents' decisions to buy, sell, or act as intermediaries are jointly determined in equilibrium. Trading by strategic agents can result in an inefficient equilibrium allocation because bilateral prices depend not only on the private valuations of all agents, but also on the share of the surplus each intermediary receives. When a trading network is not sufficiently dense, the probability that the equilibrium allocation is always efficient tends to zero as the network size increases. I derive an analytical solution for the expected welfare loss and show that it can increase with the density of the trading network. I apply this theoretical framework to show that a large interconnected financial institution can improve efficiency even after accounting for a moral hazard cost due to a possible ex-post bailout. This welfare gain should be considered when deciding whether large financial institutions are too interconnected to exist.

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1 Introduction

Banks, hedge funds, and firms trade financial assets in over-the-counter (OTC) markets to allocate risks and liquidity in the economy.[1] The conventional wisdom is that if every trader sells a financial asset at the best available price, the related risk will be allocated to institutions that are best able to bear it.[2] I show that when prices for a good in intermediated trades are set via bargaining, an efficient allocation is not guaranteed.[3] The inefficiency happens because prices depend not only on fundamental valuations, but also on the share of surplus each intermediary receives. Therefore, a sequence of bilateral trades can result in inefficient equilibrium allocation, even if bargaining in each trade is efficient. I refer to this problem as a *bargaining friction* in a decentralized exchange economy.

The bargaining friction affects the efficiency of resource allocation in OTC markets when (1) intermediation is required, (2) each intermediary cannot extract the full surplus in each trade, and (3) bilateral prices are not contingent on the endowment or prices in other trades.

These conditions are likely to hold in OTC markets. First, intermediation is at the core of OTC markets.[4] Second, prices in these markets are set by bilateral bargaining or auctions, which in general do not provide sellers the full surplus from trade.[5] Third, prices in OTC markets are not contingent on the original seller. When a seller asks a buyer for a quote, the buyer does not know whether the seller is selling a security he owns or a security someone else is trying to sell him.

The example in Figure 1 illustrates the bargaining friction. Assume A owns a financial security that he values at 0.4. He can sell it to one of two trading partners: B, who values it at 0, or C, who values it at 0.4. Based on these valuations, trade between A and his trading partners produces no gains. A gain exists, however, because B can resell the security to

[1]At the end of 2009, the outstanding gross notional of OTC derivatives was more than \$600 trillion and of bonds more than \$90 trillion (BIS Quarterly Review, June 2010, `http://tinyurl.com/25f46pp`).

[2]For example, see a speech by Andrew G Haldane, an executive director of financial stability in the Bank of England. Haldane (2009), p. 7.

[3]A good can be a loan, an insurance contract, a real asset, or a consumption good.

[4]See, for example, Bech and Atalay (2010), who provide a description of the federal funds market. Craig and Peter (2009) provide evidence for a core-periphery structure in the German banking system.

[5]Saunders, Srinivasan, and Walter (2002) document that the trading mechanism in the OTC corporate bond markets closely resembles a first-price sealed bid auction. In section 5.2, I show that a sequence of efficient auctions can result in an inefficient equilibrium allocation.

$B1$, who values the security at 1, and C can resell it to $C1$, who values the security at 0.8. I assume A cannot trade with $B1$ or $C1$ directly due to large transaction costs. All agents know the valuations and trading opportunities of each agent. B can sell the security to $B1$ for 0.5, getting half of the surplus.[6] C can sell the security to $C1$ for 0.6, also getting half of the surplus. Given that the resale value of C is higher than the resale value of B, A will sell the security to C for 0.5, which provides A with half of the surplus in this trade. The equilibrium allocation is inefficient because $B1$ does not buy the security in equilibrium even though he values it the most. In this example, each bilateral trade is efficient, but the sequence of trades is not. The outcome of the sequential trades is inefficient because B's private valuation is smaller than C's. As a result, B has a lower outside option and consequently a smaller resale value than C, who buys the security in equilibrium and sells it to $C1$.

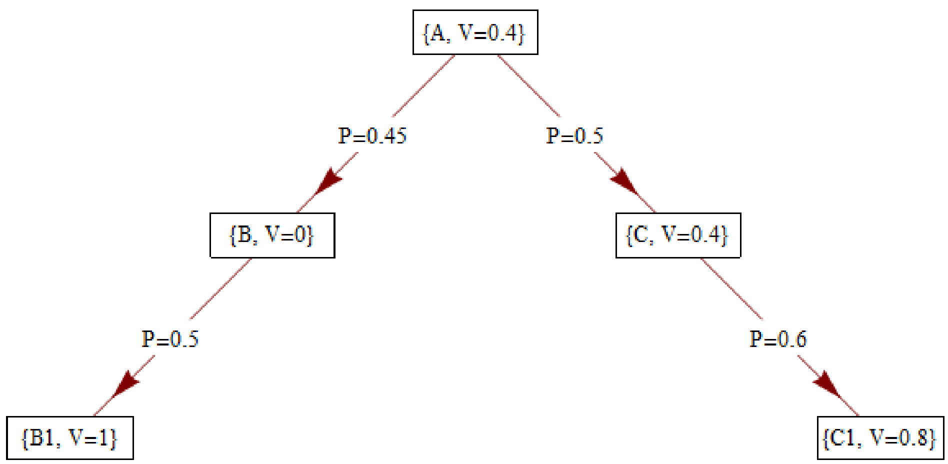

Figure 1: Inefficient Allocation with Sequential Bargaining

To study the friction in OTC markets, I develop a model to capture their main features: (1) a large number of traders, (2) prices are set by bilateral bargaining, (3) some agents can trade only via intermediaries whereas others can trade directly, and (4) surplus depends not only on the private valuation of the buyer but also on his resale opportunities. I use the model to characterize conditions when bilateral trading results in an inefficient allocation and to quantify the welfare loss. The model also allows me to analyze the relationship

[6] In this example, I assume a seller and a buyer equally split the surplus, which is defined as the difference between the buyer's reservation price and the seller's private valuation. Later I show that the bargaining friction exists in a more general setting.

between the efficiency of trading and the market structure. In addition, the model provides a new framework to study the role of large interconnected institutions in OTC markets.

In the model, agents can trade only if they have a trading relationship. Trading relationships between agents constitute a trading network that allows trading of financial assets between any pair of agents directly or using intermediaries.[7] The trading network might not include all possible trading relationships because of the high transaction costs for some agents to trade directly with each other.[8] The price in each trade is set via bilateral bargaining over the trade surplus, which depends on the resale opportunities of the buyer. Therefore, the trading decisions and bilateral prices are jointly determined in equilibrium. I develop an algorithm to compute equilibrium prices and trading decisions for any trading network and show the equilibrium is generically unique.

I use the model to characterize the trading networks in which bilateral trading always results in efficient equilibrium allocation. Trading in a complete network, in which all possible trading relationships are present, results in efficient allocation, regardless of the endowment, valuations, or the share of surplus each seller receives. The negative result is that a complete network is the only trading network that has this property. Moreover, when each seller cannot extract the full surplus in each trade and trading requires intermediation, the equilibrium allocation is inefficient for some valuations and endowments. I compute analytically prices and an equilibrium allocation for any initial allocation in a *simple economy* in which all sellers receive the same share of surplus and one agent has the highest private valuation for a good, one agent has the second-highest valuation, while all other agents have zero private valuation for the good. I derive a threshold on the maximum number of intermediaries in a trading network such that an equilibrium is efficient for all possible endowments. To study the relationship between trading efficiency in a simple economy and a market structure, I use a *homogeneous trading network*, a network in which every trading relationship is equally likely to exist. I show that if the number of trading relationships does not grow fast enough with the size of a homogeneous trading network, the probability that trading will result in the first-best allocation for all initial allocations

[7]Allen and Babus (2008) provide a survey of the literature that uses networks to analyze financial systems, but the main focus of this literature is on contagion rather than trading efficiency. Jackson (2008) provides a more general overview of network-based models in economics.

[8]Transaction costs include monitoring costs, counterparty risk, cost to enforce contracts, and so forth. For example, a seller might prefer to trade with institutions with which he has a netting agreement because doing so reduces the cost of managing the counterparty risk. Different geographical locations, currency, time zones, languages, and regulatory restrictions result in variation in transaction costs between agents.

tends to zero.

The bargaining friction can result in a substantial welfare loss, the difference between the first-best allocation and the equilibrium allocation. To examine the relationship between the welfare loss and the structure of the trading network, I compute the expected welfare loss in a homogeneous trading network for the simple economy of an arbitrary size. The expected welfare loss accounts for the uncertainty in the number of intermediaries required for trade, as well as uncertainty in the endowment and valuation processes. I find the expected welfare loss is non-monotonic in the density of the trading network. For example, randomly adding 14 million trading relationships in a market with 10,000 traders can increase the expected welfare loss by more than 400%. This result is important for any policy that affects the number of trading relationships in OTC markets. For example, a central counterparty (CCP) can decrease the cost of direct trading between some agents in the market, but if some trades still require intermediation then establishing a CCP can decrease welfare.

I apply this theoretical framework to characterize when a large interconnected financial institution that can trade with all market participants improves efficiency relative to the homogeneous trading network. I provide a threshold on the average number of trading partners in the homogeneous trading network below which having a large intermediary is beneficial because it decreases the absolute number of intermediaries between any pair of agents. I also characterize when adding more trading relationships between agents in a simple economy with one large intermediary decreases welfare. It happens because the efficiency of a financial architecture depends not only on the absolute number of intermediaries required for trade between any pair of agents, but also on the relative number of intermediaries: the number of intermediaries required to facilitate trade between the initial seller and the agent with the highest valuation relative to the number of intermediaries required to facilitate trade between the initial seller and the agent with the second-highest valuation. The social cost of a large interconnected financial institution results from an inefficient accumulation of assets on its balance sheet if it anticipates to being bailed out in some future states of the world because of its systemic importance. I model the effect of the ex-post bailout in a reduced form, assuming it increases the private valuation of the large interconnected institution ex-ante. To measure the net effect of a large interconnected financial institution on the trading efficiency, I compare the expected welfare loss in different financial architectures for different sizes of the ex-ante distortion in the private valuation. I find that a large interconnected financial institution improves trading efficiency even when the distortion in its private valuation is large. The policy implication of my analysis is that the welfare gains from large interconnected financial intermediaries should be taken into account when (re)designing of financial systems.[9]

Overall my analysis has the following implications. First, prices in OTC markets do not reflect private valuations of the buyers because prices depend also on the shares of surplus sellers receive. Moreover, even the final price in a sequence of trades of an asset may not be based on the highest valuation in the market. Second, trading efficiency depends on the market structure, which is an important consideration for policies that affect the number of trading relationships or the costs of establishing and maintaining trading relationships. Third, financial institutions with many trading partners can improve trading efficiency. Therefore, breaking down these institutions can result in an efficiency loss.

The main contribution of this paper is to introduce the bargaining friction and to study how it affects the efficiency of resource allocation in large markets. Gale and Kariv (2007) model the financial system as a network and study whether the equilibrium allocation is efficient in an environment with complete information. They show that when a seller can make take-it-or-leave-it offers to his trading partners in a financial network, the equilibrium allocation is efficient. Condorelli (2009) considers a trading in a network with many agents who have two possible valuations for a single good, sellers can make take-it-or-leave-it offers, and in which trading stops at an exogenously given time. He reaches the same conclusion as Gale and Kariv (2007)—that in a complete information environment, the equilibrium is always efficient, the initial seller extracts the full surplus, and intermediaries make zero profits. In the environment with asymmetric information, Condorelli finds the equilibrium allocation can be inefficient either because trading is terminated too early or because some sellers are too aggressive with their offers. Blume, Easley, Kleinberg, and Tardos (2009) find that in a trading network with exogenously defined intermediaries who make take-it-or-leave-it offers to a buyer and a seller by setting bid-ask prices, the equilibrium of a complete information game is efficient. I show that in case of complete information, the result in the above papers—that the equilibrium allocation is always efficient regardless of the network structure—depends crucially on the assumption that each seller extracts the full surplus in each trade.[10] I find that when surplus is divided in each trade, efficiency is

[9]For example, my analysis contributes to answering the key questions in Section 123 of the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010 that requests the Financial Stability Oversight Council to carry out a study of the economic impact of possible financial services regulatory limitations intended to reduce systemic risk.

[10]This assumption simplifies the analysis, but it is not an outcome predicted by a non-cooperative or

not guaranteed.

In an incomplete-information environment, I show that a sequence of second-price sealed-bid auctions can be inefficient even though each individual auction is ex-post efficient.[11] The intuition for this result is that when a seller auctions the good, his trading partners' bids depend on the resale price, which depends on the second-highest bid they receive when they auction the good. Given that the private valuations of other bidders, rather than those of the buyers, in each resale auction determine the resale price, the final buyer may not have the highest private value for the good among all agents in the economy.

My paper is also related to a literature that studies efficiency of networked markets. Nava (2008) studies quantity competition with intermediation in trading networks. He finds the equilibrium allocation can be inefficient even when a trading network allows all agents to trade directly. Abreu and Manea (2010) study bargaining in a network without intermediation. The equilibrium allocation can be inefficient in their model because agents maximize their profits and not the total profit created by all agents, which depends on the sequence of individual trades. Kranton and Minehart (2001) find that the equilibrium allocation is efficient in buyer-seller networks without intermediation when sellers hold ascending-bid auctions. I show that this result does not hold in more general trading networks with endogenous intermediation.

Duffie, Garleanu, and Pedersen (2005) use a search-based model of OTC markets to show that with bilateral bargaining, financial intermediaries can search too intensively for trading opportunities compared to the social optimum. In their model, intermediation is exogenous, and one intermediary is sufficient to create the maximum surplus. In my model, intermediation is endogenous and an equilibrium trading path can require more than one intermediary. In a recent paper, Wong and Wright (2011) show that a holdup problem can arise in a search-based model similar to the bargaining friction in my paper. In their setting, agents are ordered in a line and each agent can sell a good only in one direction. The authors characterize the terms of trade, how many intermediaries get involved, and

cooperative approach to bilateral bargaining. When only one seller and one buyer bargain over a surplus, the equilibrium is efficient in Rubinstein's (1982) non-cooperative game of alternating offers, but a buyer and a seller split a surplus based on their time preference parameters. Nash's (1953) axiomatic approach to bilateral bargaining assumes the outcome is efficient and determines a division of surplus based on the buyer's and seller's risk aversion.

[11]Vickrey (1961) introduced this auction and showed the optimal bidding strategy of each bidder is to bid his private valuation. If the reservation price of the seller is below the highest private valuation, the auction is ex-post efficient.

show how bubbles can emerge in the value of inventories. Using a network-based approach allows me to model general market structures in which agents can trade with many buyers and to compare different market structures in terms of their trading efficiency.

The structure of the paper is as follows. In the next section, I introduce the economic environment and equilibrium properties. I discuss the efficiency of bilateral bargaining and the welfare loss due to the bargaining friction in section 3. I analyze the benefits and the costs of large interconnected financial institutions in section 4. In section 5, I extend the analysis to additional trading mechanisms. I conclude in section 6. All proofs appear in the Appendix.

2 The Model

This section describes a model of a decentralized exchange economy with intermediation. I divide the discussion into three subsections: the economic environment, the trading mechanism, and the equilibrium analysis.

2.1 Economic Environment

There are n agents in the economy described by a set $N = \{1, ..., n\}$. A trading network is represented by a graph g, which is a set of trading relationships between pairs of agents. If a trading relationship exists between agent i and agent j then $\{i, j\} \in g$ (or $ij \in g$); otherwise, $\{i, j\} \notin g$.[12] I assume g is a *connected network*, such that every pair of agents can trade directly or using intermediaries. The trading network is *complete* if all agents can trade directly with each other. The trading network is *incomplete* if at least one pair of agents does not have a trading relationship because of high transaction costs, such as monitoring costs or counterparty risk.

There is a single good in the economy and agents cannot create new goods. An endowment vector $E = \{E_1, ..., E_N\}$ describes the endowment of the good, such that $E_i = 1$ if i owns the good in the initial allocation and $E_i = 0$ otherwise.

Agents have private valuations for the good represented by a valuations vector $V =$

[12] I assume every agent can always keep the good ($\{i, i\} \in g$ for all i), and that the trading network is undirected (if $\{i, j\} \in g$, then $\{j, i\} \in g$).

$\{V_1, ..., V_N\} \in [0,1]^n$, where $V_i \in [0,1]$ is i's private valuation for the good if he owns it.

One interpretation for different private valuations for a real asset, such as an art work, is a heterogeneity in tastes. The heterogeneity in liquidity needs or hedging demands introduces variation in private valuations for financial assets. For example, one bank has a higher willingness to pay for insurance against defaults of firms in the car industry than another bank because the loan portfolio of the first bank has a higher exposure to this industry.

The economic environment and the rationality of all agents are common knowledge among the agents.

2.2 Trading Mechanism

A trading protocol specifies how agents trade and how prices are formed. "The function of a resource allocation mechanism is to guide the economic agents (producers, consumers, bankers, and others) in decisions that determine the flow of resources" (Hurwicz 1973, p. 16). Consistent with this definition, we can think about trading as a resource allocation mechanism, because agents' trading decisions determine the equilibrium sequence of trades and the equilibrium allocation. Price formation is the heart of the resource allocation mechanism because prices govern agents' trading decisions.

Bilateral bargaining is a convenient way to model the price formation process in an OTC market. The surplus in each trade is equal to the buyer's valuation for the good minus the private valuation of the seller. I model the bargaining process in a reduced form, where each seller gets a share of the surplus equal to his bargaining ability $B_i \in (0,1)$. I assume an agent receives the same share of surplus if he sells to any one of his trading partners.[13] Therefore, any buyer from seller i receives $1 - B_i$ share of the surplus from trade between the two.

The bargaining ability vector $B = \{B_1, ..., B_N\} \in (0,1)^N$ is a vector of bargaining abilities of all sellers. Price in each trade equals the private valuation of the seller plus his share of the trade surplus, which is determined by his bargaining ability. The value of the good to the buyer depends on the value of the good to his trading partners. Therefore, the

[13] Further analysis can be generalized to the case in which the seller's share of surplus varies with the buyer's identity. When the bargaining ability is seller specific, the seller will always sell to the buyer with the highest valuation.

trading decisions of all agents are interconnected.

The price formation process I assume ensures that (1) a seller never sells for a price below his private valuation, (2) a buyer never pays a price above the maximum between his private valuation and his resale value, and (3) if a seller decides to sell a good, he always sells it to the trading partner with the highest valuation. In further analysis, I characterize for what vectors of bargaining ability trading will be efficient.

We can think about the price-setting mechanism as Nash bargaining in which the outside option of the seller equals his private valuation, and the outside option of the buyer is zero. In section 5, I consider a price-setting mechanism in which the seller's outside option is the maximum between his private valuation and the price he receives when he sells to another trading partner. In this case, the outside option of the buyer can be positive because if the seller sells to some other trading partner, the buyer can still buy the good via intermediaries when a trading network has loops. The bargaining friction exists in this case as well, but the computation of the equilibrium is more complicated.

Trading is sequential; each agent, who holds the good either keeps it or sells it to one of his trading partners. Trading continues until the current owner prefers to keep the good.

2.3 Equilibrium and Equilibrium Properties

In equilibrium, each agent bargains and sells the good to one of his trading partners who pays the highest price, if this price is above his private valuation. Let $\sigma_i \in N(i,g) \cup i$ be an *equilibrium trading decision* of agent i if he has the good, where $N(i,g) = \{j \in N\} \mid ij \in g\}$ is the set of trading partners of i in a trading network g. The *equilibrium valuation* of agent i, P_i, equals his private valuation, if he keeps the good in equilibrium. If he sells the good in equilibrium then P_i equals to the price he gets. Next I formally define the equilibrium trading decisions and valuations.

Definition (Equilibrium)**.** *Equilibrium trading decisions and valuations are defined as follows:*

i. For all $i \in N$, agent i's equilibrium valuation is given by:

$$P_i = \max\{V_i, \max_{j \in N(i,g)} V_i + B_i(P_j - V_i)\}.$$

ii. For all $i \in N$, agent i's equilibrium trading decision is given by:

$$\sigma_i = \arg\max_{j \in N(i,g) \cup i} P_j.$$

If agent i keeps the good in equilibrium then $\sigma_i = i$ and his valuation for the good is his private valuation: $P_i = V_i$. If j has the highest valuation for the good among all trading partners of i and this valuation is higher than i private valuation then i sells to j in equilibrium, $\sigma_i = j$. The *equilibrium bilateral price* between i and j, $P(i,j) = (1 - B_i)V_i + B_i P_j$, determines the equilibrium valuation of i, P_i, for the good.

The definition of equilibrium is different from the Nash equilibrium or subgame-perfect Nash equilibrium for two reasons. First, agents make trading decisions sequentially, which means I cannot use Nash equilibrium as an equilibrium concept. Second, even though trading decisions are sequential, when a trading network is not a directed tree I cannot use backward induction to solve for the subgame-perfect equilibrium because there are no exogenously given final buyers. In equilibrium as defined above, bilateral prices and agents' decisions to buy, sell, or act as intermediaries are jointly determined even though trading is sequential.[14]

I provide an algorithm to compute equilibrium prices and trading decisions in any trading network in section 7.1 of the Appendix. The idea is to solve for the equilibrium trading decision of each agent sequentially, starting with the agent who has the highest private valuation for the good. Since the agent with the highest private valuation will always keep the good in equilibrium, we can consider the trading decision of the agent with the second-highest private valuation, and so on, until we determine equilibrium valuations and trading decisions of all agents. I refer to this solution algorithm as recursive backward induction because I use backward induction recursively to solve for the optimal trading decision of agent i conditional on optimal trading decisions of all agents with higher private valuation than i's.[15]

[14]Off-equilibrium trading decisions are not well defined when the trading network is not a directed tree. For example, consider an equilibrium in which A sells to B and B decides to keep the good. If B would want to consider a deviation from keeping the good to selling it to A, holding A's equilibrium trading decision fixed, then the payoff of B from this deviation is not determined, because A sells to B and B sells to A.

[15]In section 7.2 I provide an alternative solution algorithm. Specifically, I show that the trading mechanism is a contraction mapping and therefore according to the contraction mapping theorem one can solve for equilibrium valuations by the iterative approach that I describe in section 7.2. The benefit of this solu-

In intermediated trades, the surplus depends on the resale prices, which in turn depend on the bargaining abilities and on the private valuations of all agents in the economy. Equilibrium prices and trading decisions determine equilibrium gains from trade and an equilibrium allocation path—the sequence of trades from the agent with the endowment to the final buyer.

The equilibrium gain from trade of agent i is given by:

$$S_i = \begin{cases} P_i - V_i & \text{if } i \text{ has the initial endowment (seller)} \\ P(i,j) - P(k,i) & \text{if } i \text{ buys from } k \text{ and sells to } j \text{ (intermediary)} \\ V_i - P(k,i) & \text{if } i \text{ buys the good from } k \text{ and does not sell (buyer)} \\ 0 & \text{otherwise.} \end{cases}$$

Prices are increasing along the equilibrium trading path because an intermediary never buys a good for a higher price than the price for which he sells it for. Next, I prove the equilibrium properties.

Proposition 1. *There are no trading cycles in equilibrium.*

This proposition means that in equilibrium, for any endowment, there is a finite trading path that starts with the agent endowed with the good and ends with the final buyer. The number of trades in equilibrium is equal to the number of intermediaries plus one. In equilibrium, a buyer never sells the good to an agent from whom he previously bought the good.

I use the solution algorithm to prove the uniqueness of the equilibrium.

Proposition 2. *Equilibrium valuations are unique and equilibrium trading decisions are generically unique.*

The intuition for the proof is that each agent has an optimal trading decision, conditional on the optimal trading decisions of all agents with a higher valuation than his. Therefore, the solution algorithm allows us to solve for the unique equilibrium recursively. The formal proof in the Appendix verifies the uniqueness does not depend on the algorithm. The equilibrium trading decisions are *generically* unique because some agents can

tion algorithm relative to the recursive backward induction is that it allows us to solve fast for equilibrium in large trading networks.

be indifferent between selling to several buyers. However, if private valuations are drawn from a continuous distribution, then the probability that any agent is indifferent is zero and the equilibrium is unique. If trading decisions are unique then for each endowment, there is a unique sequence of bilateral trades between the initial seller and the final buyer.[16] The uniqueness of the equilibrium allows me to study the efficiency of the final allocation in different economic environments without a need to choose which one of the equilibriums I compare to the socially optimal allocation.

3 Trading Efficiency

In this section, I discuss efficiency of the equilibrium allocation and compute the expected welfare loss due to the bargaining friction.

During the trading process, the good is transferred between the agents. Allocation vector $a(g, E, t) = \{a_1^t, \ldots, a_n^t\}$ specifies which agent has the good after t trades, such that if $a_i^t = 1$ then i has the good after t trades. The initial allocation is the endowment, $a(g, E, 0) = E$, and if the trading ends after T trades then $a(g, E, T)$ is the equilibrium allocation. If the trading network is connected then all allocations are feasible, such that any agent can be the final buyer or the intermediary.

An allocation $a(g, E, t)$ is *(Pareto) efficient* if the agent who owns the good in this allocation has the highest private valuation among all agents, or no other agent has a strictly higher valuation. The question I address next is when the equilibrium allocation is efficient.

Proposition 3. *(i) If g is complete then the equilibrium allocation is efficient for any B, E, and V.*

(ii) If g is incomplete then for any B, there exist vectors E and V such that the equilibrium allocation is inefficient.

Proposition 3 states that when intermediation is required, efficiency is not guaranteed. If g is complete then in equilibrium an agent with the endowment sells directly to the agent with the highest private valuation.[17] Given that each bilateral bargaining is efficient, the equilibrium allocation is efficient. When g is incomplete, exists a pair of agents that

[16] If the initial seller decides to keep the good in equilibrium then no trading will happen in equilibrium.

[17] If bargaining ability of a seller i depends on the identity of the buyer then a number of trades can occur

requires at least one intermediary to trade. If this intermediary cannot extract the full surplus when he resells the good then his resale price is below the private valuation of the buyer. If the private valuation of the seller is also above the resale price, no trade occurs even if the final buyer has a higher private valuation than the seller's.

Proposition 3 opens a Pandora's box, because it states that trading in any decentralized market with intermediation can result in an inefficient allocation of resources. It invites a wide range of questions related to welfare, normative and positive analyses of a decentralized economy with intermediation in general and OTC markets in particular. How likely an equilibrium to be efficient and what is the expected welfare cost due to the bargaining friction? What financial architecture is socially optimal? Can the bargaining friction explain the observed characteristics of OTC markets? In the rest of the paper, I focus on the welfare and normative analyses, and leave the positive analysis for future research.

Next, I show that when a trading network is incomplete, adding a new trading relationship can decrease efficiency. In Figure 2, five agents $\{A, B, C, D, E\}$ trade on a network with four trading relationships (solid lines). A private valuation of each agent appears after the comma next to his name. The bargaining ability of all agents is 0.5, such that any seller and buyer receive an equal share of the surplus from trade. I assume C has the endowment and show that adding a trading relationship (dashed line) decreases social welfare. When C has the good, he sells it to A, who has the highest valuation, through B when only four trading relationships are present. C sells the good to B for 0.25 and not to D for 0.15, because B's resale value is higher than D's. If C could sell to E directly then the price would be 0.3, which is higher than 0.25. Therefore, C would sell to E in this case and the allocation would be inefficient.

The intuition for the result is that when C trades with A and E via intermediaries, the share of the surplus C receives depends not only on his bargaining ability but also on the bargaining ability and outside options of the intermediaries. Adding a trading relationship between C and E increases the share of the surplus that C receives from trading with E, whereas the share of surplus C receives from the intermediated trade with A does not change. In general, the surplus from trade between any seller and another agent in the trading network, not necessarily a trading partner, depends on the number of intermediaries, their bargaining abilities, and their outside options. Changes in the network structure can change the surplus shares and consequently the equilibrium allocation.

in equilibrium even when a trading network is complete. However, the equilibrium allocation is always efficient in this case as well.

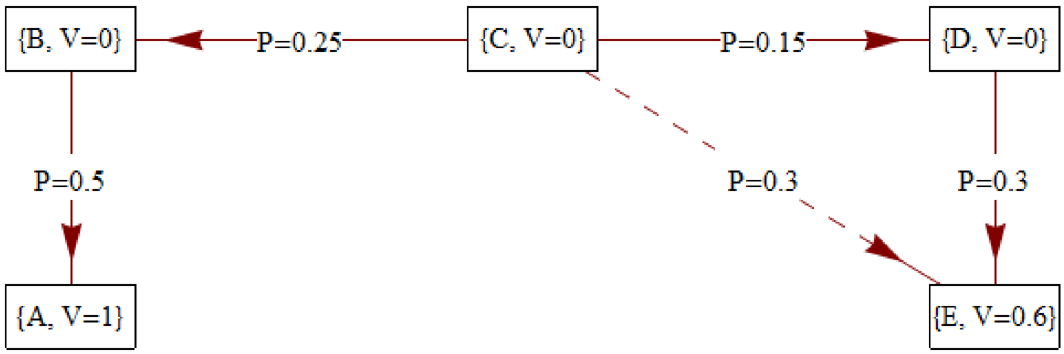

Figure 2: Adding Link $C \to E$ Decreases Efficiency

To study the relationship between the density of the trading network and trading efficiency in a large market, I further study a simple economy.

Definition (Simple Economy)**.** *In a simple economy, all agents have the same bargaining ability* $b \in (0,1)$*,* $n-2$ *agents have a private valuation equal to zero, one agent has a valuation equal to* 1*, and one agent has a valuation* v*, where* $0 < v < 1$*.*[18]

Let $\mathcal{V}^s$ represent the set of all private valuation vectors that satisfy the restrictions of the simple economy, and let $B^s = \{b, \ldots, b\}$ represent the vector of bargaining abilities in the simple economy. The above assumptions allow me to focus on the relationship between the amount of intermediation in the market and its efficiency. Prices in the simple economy depend on the number of intermediaries and not on the outside options or bargaining ability of the intermediaries. This simplification still allows us to study efficiency of large markets with substantial variation in the amount of intermediaries required to facilitate different trades.

In the simple economy, the equilibrium allocation will be efficient for any initial allocation if the agent with the second-highest valuation does not keep the good in equilibrium.[19] If he sells the good then the agent with the highest valuation will be the final buyer, because all other agents have valuations smaller than v. Agents with private valuations equal

[18]The results can be generalized to an economy in which $n-2$ agents have the same private valuation, which is below the second-highest private valuation.

[19]When more than three types of private valuations exist, the number of possible inefficient equilibrium outcomes is larger because any agent, except the one with the lowest valuation, can keep the good in equilibrium.

to zero will sell the good, because prices are above their private valuations. When the equilibrium allocation is inefficient, the welfare loss is the difference between the highest and the second-highest valuation: $1 - v$.

In an economy with shocks to endowments and valuations, the agent with the second-highest valuation can sometimes sell directly to the agent with the highest valuation and the equilibrium allocation is always efficient. However, it is unlikely always to be the case in a large market. The question I address next is what the maximum number of intermediaries between these two agents can be such that the agent with the second-highest valuation does not keep the good in equilibrium.

The answer to this question depends on the second-highest valuation (v) and the bargaining ability of intermediaries (b). The more intermediaries required to facilitate trade between the agent with the second-highest valuation and the agent with the highest valuation, the smaller the resale price of the first intermediary who buys from the initial seller. If only one intermediary is required then his resale value is b. If two intermediaries are required then the resale value of the first intermediary is b^2, and so on. Given that sellers cannot extract the full surplus in each trade, a "leakage" of surplus happens each time another intermediary is required to resell the asset, because the resale value is always below the equilibrium valuation of the buyer.[20]

In the next proposition I derive a simple condition on the network structure to ensure that bilateral bargaining always results in an efficient equilibrium allocation.

Proposition 4. *In the simple economy, the equilibrium allocation is efficient for all E and $V \in \mathcal{V}^s$ if and only if any pair of agents requires at most $\hat{d} = \left\lfloor \frac{\log(v)}{\log(b)} \right\rfloor$ intermediaries to trade in a trading network g.*[21]

The intuition for this result is that we need to consider the worst-case scenario in which the agent with the second-highest valuation gets the endowment and requires the maximum number of intermediaries to sell the asset to the efficient buyer. If the efficient trade is executed in this case then trading will be efficient for any realization of endowment or valuations in the trading network.

[20] A similar "leakage" of surplus happens in a general economy, but the resale price of the first intermediary depends not only on the number of intermediaries required to facilitate trade but also on their private valuations and bargaining abilities.

[21] The floor function rounds down to the closest integer. For example, if $\frac{\log(v)}{\log(b)} = 2.2$ then $\left\lfloor \frac{\log(v)}{\log(b)} \right\rfloor = 2$.

How likely are markets with thousands of agents and trading relationships to be always efficient? How large is the welfare loss in large markets? Finally, are networks with more trading relationships more likely to facilitate an efficient trade and decrease the (expected) welfare loss? To address these questions, we need a simple model of a large complex market. I introduce this model in the next section.

3.1 Homogeneous Trading Networks

To model complex trading networks in which agents can have a different number of trading partners, I assume the probability of each trading relationship is $0 < p < 1$. I refer to p as a *degree of network completeness* because it describes what percentage of all possible trading relationships are established. When $p = 1$, the trading network is complete and the equilibrium is always efficient (Proposition 3). As p decreases, the amount of intermediation required to facilitate trades in this market increases. When $p = 0$, the economy is an autarky: agents cannot trade with each other. Further analysis focuses on markets that are neither complete networks nor autarky, which is the case with OTC markets. The goal of the further analysis is to show that (1) the probability of inefficient allocation can be substantial even when each agent can trade directly with many other agents, (2) more interconnected markets are not necessarily more efficient, (3) a financial architecture with one agent who can trade with all market participants can be more beneficial to society than a financial architecture in which all agents can trade with a small number of counterparties.

A trading network in which each trading relationship is equally likely to exist is referred as a homogeneous trading network or $\tilde{g}^h$, also called a Erdős-Rényi random network.[22] Erdős and Rényi (1961) first studied this network and established an important result for the threshold probability: if $p(n)$ grows faster than $\frac{\log(n)}{n}$ then the probability that the homogeneous trading network is connected tends to one as the network size increases.[23]

In the next proposition, I compute the probability that bilateral bargaining always attains an efficient equilibrium allocation in a simple economy.[24] Then I provide a threshold

[22] I use this definition because from the ex-ante perspective, each agent has the same (binomial) distribution of the number of trading partners. Albert, Jeong, and Barabási (2000) attribute this trading network to the class of homogeneous random graphs based on this characteristic.

[23] The notation $p(n)$ specifies that p can depend on the number of market participants (n).

[24] For these results, I maintain the assumption of a simple economy. To compute the probability of inefficient allocation for any vector of bargaining abilities and valuations one needs to use the solution algorithm in section 7.2 to solve for equilibrium outcomes in many instances of an economy.

on p, below which this probability tends to zero with the size of the network (Theorem 1).

Proposition 5. *In the simple economy, the probability that the equilibrium allocation is efficient for all E is at most $1-(1-p)^{K^{\hat{d}}}$, where $\hat{d} = \left\lfloor \frac{\log(v)}{\log(b)} \right\rfloor$, and $K=(n-1)p$ is the expected number of trading partners of each agent.*

The intuition for the proof is as follows. The equilibrium allocation is efficient for all E if the agent with the second-highest valuation sells the good to the agent with the highest valuation. According to Proposition 4, the agent with the second-highest valuation sells the good to the agent with the highest valuation if not more than $\hat{d}$ intermediaries are required to facilitate trade between these agents.[25]. Therefore, the probability that the equilibrium is efficient for all endowments is the probability that the agent with the second-highest valuation requires less than $\hat{d}+1$ intermediaries to sell the good to the agent with the highest valuation. With probability p, the two agents can trade directly. Therefore, with probability $1-p$, they will require at least one intermediary to facilitate the trade. They will require at least two intermediaries if all trading partners of the agent with the second-highest valuation require at least one intermediary. In the proof for proposition 5, I derive a lower bound on the probability for any pair of agents in a homogeneous trading network to require at least d intermediaries to trade and use this new result for this type of random networks to compute an upper bound on the probability that the equilibrium allocation is always efficient.

The following theorem establishes a relationship between the degree of network completeness and the efficiency of trading in large trading networks.

Theorem 1. *If the degree of completeness of $\tilde{g}^h$ is smaller than $\hat{p} = O(n^{-\frac{\hat{d}}{\hat{d}+1}})$ then the probability that the equilibrium allocation is efficient for all E tends to zero as the size of the trading network increases.*

The above theorem provides a threshold on the growth rate in the average number of trading partners of each agent such that if this growth rate is smaller than the order of

[25] Even if the probability that the agent with the second-highest valuation gets the endowment is zero, the probability of the inefficient allocation and the expected welfare loss can be substantial, because if the agent with the second-highest valuation keeps the good in equilibrium then some fraction of agents with zero valuation will sell to this agent directly or indirectly if they get an endowment. I compute the fraction of agents with zero valuation who sell in equilibrium to the agent with the highest valuation when I compute the expected welfare loss in section 3.2

$n^{1/(\hat{d}+1)}$ then the probability the equilibrium is efficient for all E tends to zero as the size of the trading network increases. For example, if the bargaining ability of each seller is smaller than the second-highest valuation ($b < v$), in which case $\hat{d} = 0$, then probability that the equilibrium allocation is efficient for all E tends to zero unless each agent can trade directly with a constant proportion of all agents in the economy.[26]

So far the analysis has showed that large and interconnected markets are not always efficient and that the probability of an inefficient allocation can be close to one. However, if we want to compare different financial architectures, we need to consider not only the probability of an inefficient allocation but also the welfare loss given the inefficient allocation. In the next section, I quantify the expected welfare loss due to the bargaining friction for different financial architectures.

3.2 Welfare loss due to the bargaining friction

I divide further analysis into three parts. First, I compute the expected welfare loss in a simple economy and the homogeneous trading network. Next, I provide comparative statics with respect to the degree of network completeness and the bargaining ability. Finally, I compare the expected welfare loss in the homogeneous trading network to that in a network with one agent who can trade with all other agents. This comparison allows me to study the consequences of limiting the size or number of trading relationships of each market participant.[27]

In a general economy n agents have different private valuations. Each agent can potentially be an initial seller or a final buyer. Therefore, n final allocations and n initial allocations are possible. Assume agents are ordered in an increasing order with respect to their valuations, such that agent n has the highest valuation and agent 1 the lowest. Let $L = \{V_n - V_1, V_n - V_2, \ldots, V_n - V_{n-1}, 0\}$ be a column vector of the welfare loss in each equilibrium allocation, where $L_i = V_n - V_i$ is a welfare loss if agent i holds the good in equilibrium.[28] Let M be a matrix of transition probabilities, such that M_{ij} is a transition

[26]When $b < v$, the agent with the second-highest valuation sells the good only if he can trade directly with the agent with the highest valuation, which happens with probability $p(n)$. Therefore, if the degree of network completeness $p(n)$ decreases with the size of the trading network ($\lim_{n\to\infty} \frac{p(n)}{n} = 0$), the probability that the equilibrium allocation is efficient for all E tends to zero with the size of the trading network.

[27]Studying the consequences of limiting the size of large financial institutions is one of the main issues addressed in section 123 of the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010.

[28]An alternative measure of welfare loss can take into account the initial endowment and not only

probability from the initial allocation, in which agent i owns the good, to the equilibrium allocation, in which agent j owns the good. The transition probabilities depend on the valuations process and the network structure. Let $Q = \{q_1, \ldots, q_n\}$ be a row vector of probabilities, such that q_i is the probability that agent i is endowed with the good. Then the *expected welfare loss* is given by

$$EWL(V, B, Q, g) = Q_{1\times n} M_{n\times n} L_{n\times 1}. \tag{1}$$

In section 5.3, I discuss how to compute numerically the expected welfare loss for any network structure, distribution of endowments, and valuations. Further, I study a simple economy with n agents but only three types of allocations, which simplifies the analysis and allows for comparative statics of complex economic environments and market structures. In section 7.3 of the Appendix, I outline the steps required to compute the expected welfare loss given by

$$EWL^h(b, n, p, Q^h, G) = \sum_{l=0}^{D^h-1} F(l{+}1|n, p) \left(q_{SB} + q_{TB} \sum_{j=1}^{D^h+2-l} f(j|n, p) F(j + l) \right) \int_{b^{l+1}}^{b^l} (1{-}v) dG(v), \tag{2}$$

where $D^h(n, p)$ is the maximum number of intermediaries in a homogeneous trading network of size n and degree of completeness p, $F(l|n, p)$ is the probability that the agent with the second-highest valuation requires at least l intermediaries to trade with the agent who has the highest private valuation for the good, $f(l|n, p) = F(l - 1|n, p) - F(l|n, p)$ is the probability that an agent with the third-highest valuation requires l trades to trade with the agent with the second-highest valuation, q_{SB} (q_{TB}) is the probability that the initial endowment is second best (third-best), and $G(v)$ is a CDF that describes the distribution of v.[29]

the equilibrium allocation. For example, if the endowment belongs to agent i then $L_i = \frac{V_n - V_{eq.|E_i=1}}{V_n - V_i}$ measures what percent of the potential surplus is lost because the resource allocation process is made by strategic agents. In this case $V_n - V_{eq.|E_i=1}$ is the difference in valuations in the first-best allocation and the equilibrium allocation, and $V_n - V_i$ is the maximum surplus possible to achieve by trading. If the equilibrium allocation is first-best than the loss is zero. This measure is well defined for all endowments that are not first-best. In a simple economy, if the endowment is third-best and the equilibrium allocation is second-best, the welfare loss according to this measure is $1 - v$, which is the same as the welfare loss according to the measure I use in the paper.

[29] I use the expected welfare loss formula (equation 2) for comparative statics (Proposition 6) and for computing the expected welfare loss in large trading networks with different degrees of network completeness. To evaluate the expected welfare loss, we need formulas for D^h and $F(l|n, p)$. For D^h, I use the

The expected welfare loss computation aims to measure the amount of inefficiency in an economy in which different trading opportunities are determined by realizations of three types of shocks. The endowment shock determines who gets the endowment, the valuations shock determines the agents' valuations for the endowment, and the third shock determines the "location" of the endowment and valuations in the trading network.[30] In other words, the endowment vector and the valuations vectors are not sufficient to determine efficiency of the resource allocation process because the structure of trading relationships in the market will determine how much intermediation is required for each realization of the endowment and valuation shocks, and the efficiency of the resource allocation process depends crucially on the amount of intermediation.

After the shocks are realized, agents trade and the final allocation is determined. For some realizations of the shocks, no trade is required because agent with the highest valuation gets the endowment. For some realizations of the shocks, agent with the endowment can sell directly to the agent with the highest valuation. However, sometimes the agent with the endowment will have the third-highest valuation and the equilibrium allocation depends on the amount of intermediation required to sell to the agent with the highest valuation directly, or via the agent with the second-highest valuation. The assumption of a homogeneous trading network allows me in a tractable way to capture the variation in the amount of intermediation in a large market in which establishing or maintaining trading relationships is costly.

Table 1 reports the expected welfare loss (Eq. 2) and the probability of inefficient allocation (given by Eq. 22 in the Appendix) for networks of different sizes and degrees of completeness.[31] In this numerical example, I assume a seller and a buyer split the surplus equally ($b = 0.5$), each agent is equally likely to get the endowment, and the second-highest

lower bound for the expected maximum number of intermediaries that I derive in Lemma 1 (section 7.3). No exact analytical solution exists for $F(l|n,p)$ for this type of random networks. I use a novel approach to derive formulas for $F(l|n,p)$ in section 7.5. These formulas can underestimate the true probability for $l > 2$. In section 5.3, I compare my computation of the expected welfare loss to the numerical solution (see Figure 9).

[30]When the economy is populated by economic agents (firms, banks, investors, consumers) who have trading relationships, such as contracts, or face various constrains for bilateral trades, such as technological and compatibility constrains, these trading relationships constitute a trading network that represents the "space" for the flow of economic activity.

[31]The difference between the expected welfare loss and the probability of inefficient allocation is that the later is not accounting for the welfare loss given inefficiency only for the probability of an inefficient outcome.

valuation is uniformly distributed between zero and one ($G(v) = v$). In the last column, I also report the expected welfare loss (Eq. 2) when the agent with the second-highest valuation receives the endowment ($q_{SB} = 1$), which is the worst-case scenario for realization of the endowment shock in a simple economy. In the third column, I report the lower bound on the expected maximum number of intermediaries between any pair of agents (D^h) for a trading network with n agents, each with K trading partners on average.[32]

The expected welfare loss and the probability of inefficient allocation reported in Table 1 are non-monotonic in the degree of network completeness. For example, in a market with 10000 agents, when each agent can trade with 200 trading partners on average, the expected welfare loss is 0.457%. Increasing the average number of trading partners to 3000, by adding randomly 14 million more trading relationships on average, results in an expected welfare loss of 1.838%, which is more than four times higher.

Table 1: **Efficiency of trading in a homogeneous trading network**

n	K	D^h	EWL^h	PIA^h	$EWL^h \| q_{SB} = 1$
100	5	3	5.6%	18.6%	27.3%
100	10	2	3.5	13.5	16.9
1000	10	3	4.2	15.3	30.4
1000	15	3	2.7	10.3	25.3
1000	30	2	3.2	12.7	18.5
1000	100	1	1	4.1	11.2
1000	300	1	1.8	7.4	8.7
10000	20	3	3.6	13.8	32.1
10000	40	3	1.7	6.5	25.8
10000	80	2	3.2	12.6	20.6
10000	100	2	3	11.9	18.1
10000	200	2	0.5	1.8	12.5
10000	3000	1	1.8	7.4	8.7

[32] I derive D^h in Lemma 1 in the Appendix. If the maximum number of intermediaries in the trading network is higher than D^h then the expected welfare loss and the probability of inefficient allocation reported in the table are the lower bounds on these quantities.

3.3 Comparative Statics

I conclude this section with a comparative statics of the expected welfare loss with respect to the degree of network completeness and bargaining ability.

From Table 1, we learn that the relationship between the degree of market completeness and the expected welfare loss is not monotonic when each agent is equally likely to get the endowment. To understand the intuition for this result, consider the example in Figure 2. Assume that with probability p, a trading relationship exists between C and E and between A and C. The expected welfare loss is maximized for $p = 0.5$ because the probability of inefficient allocation is $p(1-p)$, which is the probability that a trading relationship exists between A and E but not between A and C.[33] When $p = 0$ ($p = 1$), the expected welfare loss is zero because C requires one (zero) intermediary to trade with A and with E. Hence, the difference in the number of intermediaries between C and A versus between C and E is zero when $p = 0$ or $p = 1$. When $p = 0.5$, the probability that the relative number of intermediaries is one is the highest.

In a large homogeneous trading network, a similar effect introduces a non-monotonicity between the expected welfare loss and the degree of network completeness. The implication is that adding more trading relationships can increase the expected welfare loss. The intuition is similar to the one presented in the example. A financial architecture in which almost any randomly chosen pair of agents requires one intermediary to trade is more efficient than a financial architecture in which almost any randomly chosen pair of agents requires two intermediaries to trade, which can be more efficient than a financial architecture in which most agents require two intermediaries to trade with half of the agents and one intermediary to trade with the other half. Therefore, we cannot rank efficiency of financial architectures based solely on the degree of network completeness, because the relative number of intermediaries also matters for efficiency.

Next I study when an increase in the degree of network completeness decreases or increases trading efficiency. If the degree of network completeness is such that the maximum number of intermediaries is one then the probability of inefficient allocation (Eq. 22) reduces to $(1-b)(\frac{1}{n}(1-p) + \frac{n-2}{n}p(1-p)^2)$. If the second-highest valuation is higher than the bargaining ability ($v > b$) then with probability $1-p$, the agent with the second-highest valuation, SB, keeps the good. With probability $p(1-p)$, the agent with the third-best valuation, TB, sells the good to SB and not to the agent with the highest valuation, FB.

[33] The expected welfare loss when $p = 0.5$ is $0.5(1-0.5)(1-0.6) = 0.1$.

If the endowment is the third-best allocation then the probability of inefficient allocation is $(1-b)p(1-p)^2$, which is the highest when $p = \frac{1}{3}$. When each agent is equally likely to get the endowment, the probability that the endowment is third best is $\frac{n-2}{n}$, which is close to one in large networks.

When the maximum number of intermediaries is larger than one, then with probability $F(d)(1-F(d))$, agent TB requires at least d intermediaries to trade with agent FB and less than d intermediaries to trade with agent SB. Using the lower bound for $F(d)$ derived in the proof of proposition 5, we can approximate this probability as $(1-p)^{((n-1)p)^{d-1}}(1-(1-p)^{((n-1)p)^{d-1}})$. This probability is maximized when $(1-p)^{((n-1)p)^{d-1}} = 0.5$. The $\hat{p}(d,n)$ that solves this equation, after approximating $\log(1-p)$ with a first-order Taylor series around $p = 0$, is given by

$$\hat{p}(d,n) = \left(\frac{\log(2)}{n^{d-1}}\right)^{1/d}, \tag{3}$$

The above formula provides an approximation for the local maximums of the expected welfare loss as a function of the degree of network completeness when more than one intermediary might be required to facilitate trade between any two agents.

The next step is to characterize when the expected welfare loss increases or decreases as a function of the degrees of network completeness. Let $\check{p}(d,n)$ be the lowest p such that the maximum number of intermediaries in a trading network with n agents is d. Using Lemma 1, $\check{p}(d)$ is a solution to $\log(n)d - \log(\log(n)) = -\log(\check{p}(d,n))d - \log(-\log(1-\check{p}(d,n))$. First, I use the first-order Taylor approximation around $\check{p}(d,n) = 0$ for the right-hand side of the above equality to get $\log(n)d - \log(\log(n)) = -(1+d)\log(\check{p}(d,n)) - \check{p}(d,n)/2$. Then the solution is given by $\check{p}(d,n) = 2(1+d)ProductLog\left(\frac{\log(n)^{1/(1+d)}}{2(1+d)n^{d/(1+d)}}\right)$, where $ProductLog(z)$ is the principal solution for w in $z = we^w$. The second-order expansion of this function, $w - w^2$, gives the following approximation[34]:

$$\check{p}(d,n) = \frac{\log(n)^{1/1+d}}{n^{d/(1+d)}} - \frac{\log(n)^{2/(1+d)}}{2(1+d)n^{2d/(1+d)}}. \tag{4}$$

Figure 3 plots the expected welfare loss EWL^h (solid line) and the lower bound on the expected maximum number of intermediaries D^h (dashed horizontal line) as a function of p, assuming $n = 1000$ and $b = 0.5$. The values for $\hat{p}(d,n)$, given by Eq. 3, and $\check{p}(d,n)$, given by Eq. 4, are marked as solid and dashed vertical lines, respectively. From this

[34]The first derivation of the series was done by Euler in 1783, but he acknowledged Lambert was the first to study it in 1758. Therefore, the function is also known as the Lambert W-function.

plot, we learn the approximation for the local maximum is good. Also, the approximation for $\check{p}(d,n)$, given in Eq. 4, accurately identifies transitions from $d+1$ to d intermediaries. $\check{p}(d,n)$ is upper biased in identifying local minimums in the function of the expected welfare loss with respect to p.

The conclusion is that more trading relationships increase the expected welfare loss when the degree of network completeness is in the following range: $\check{p}(d,n) \leq p(n) < \hat{p}(d,n)$ for $d = \{1, 2, \ldots, D^h\}$. More trading relationships decrease the expected welfare loss if $p > 1/3 = \hat{p}(1,n)$ or if $\hat{p}(d,n) \leq p(n) < \check{p}(d-1,n) - \epsilon$ for $d > 1$, where ϵ corrects the bias, but the size of the correction is not available in a closed form.

The non-monotonic relationship between the degree of network completeness and the expected welfare loss will depend on the endowment process and the process for private valuations. If the agent with the second-highest valuation always gets the endowment, the relationship is monotonic. If the endowment is always third best then the relationship is non-monotonic. The numerical examples in Table 1 show that we should not assume the relationship is monotonic, especially when we consider a policy intervention that changes the amount of trading relationships or the cost to establish these trading relationships in OTC markets.

The next question I address is the effect of the bargaining ability on the expected welfare loss.

Proposition 6. *The expected welfare loss in a simple economy is decreasing in the bargaining ability of the sellers (b) for any size of a homogeneous trading network (n), degree of network completeness (p), endowment process (Q^h), and valuations process $G(v)$.*

The implication of Proposition 6 is that if each agent is equally likely to get an endowment and to have one of the three valuations, then everyone would benefit from agreeing on a trading protocol in which each seller gets a higher share of surplus. In this case, after many trading periods, each agent will get an equal part of the increase in the total welfare. However, this solution requires a substantial symmetry between the agents. If some agents are more likely to get the endowment than others, or if some agents are more likely to have the highest valuation for the good, then the incentives of all of the agents will vary, and achieving an agreement is harder.[35]

[35] At the limit, if the bargaining ability of all sellers is one (each agent can make a take-it-or-leave-it offer) then the equilibrium can include bubbles, such that all or some agents trade in a circle at a price

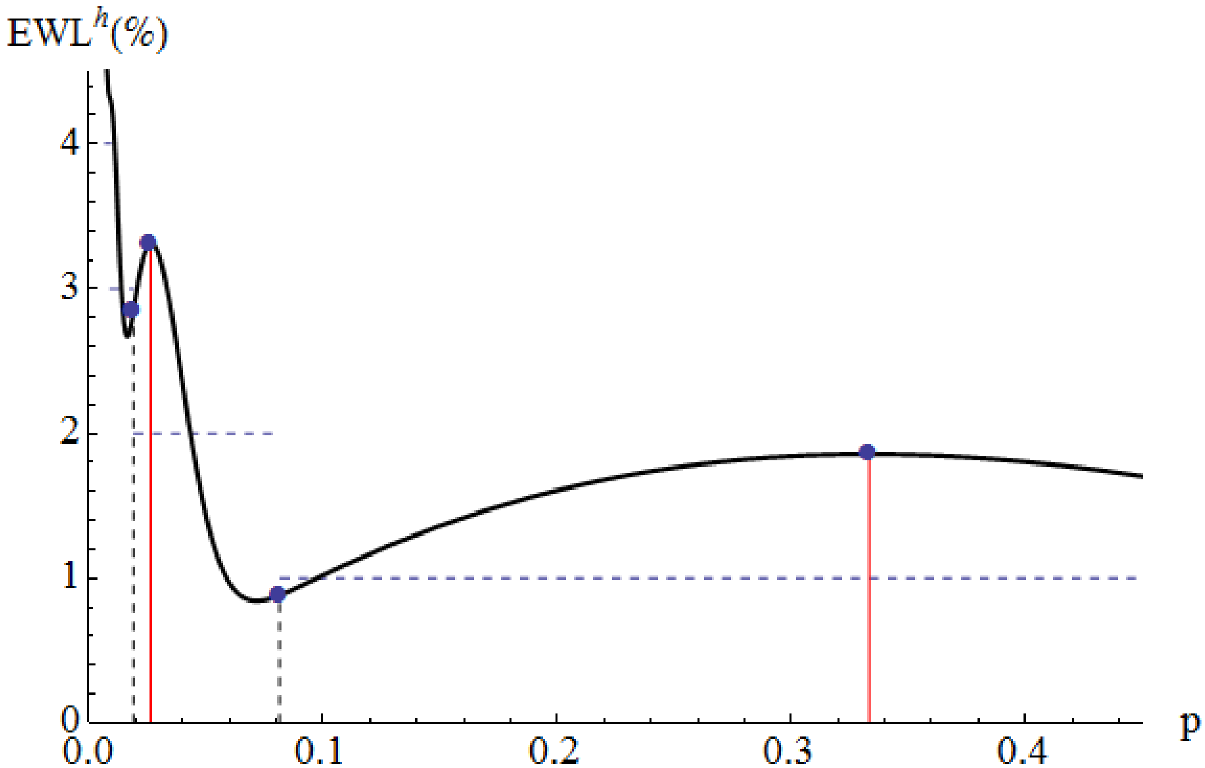

Figure 3: Homogeneous Trading Network ($n = 1000$)

In the next section, I analyze efficiency of an alternative financial architecture and study the role of large interconnected financial institutions. In this analysis, I use a normative approach to compare different financial architecture in terms of their trading efficiency.

4 Financial architecture with one large intermediary

In this section, I study a simple economy with one agent who can trade with all other agents directly, and other agents can trade with each other directly with probability p (see Figure 4). I refer to this financial architecture as an *inhomogeneous trading network* or $\tilde{g}^{ih}(n,p)$.[36] If we interpret this agent as a large interconnected financial institution, then comparing the expected welfare loss in homogeneous and inhomogeneous trading networks allows us

above the highest private valuation. However, if agents discount future payoffs or some fixed transaction costs are present then this type of equilibrium can be ruled out.

[36]This trading network belongs to a class of inhomogeneous random graphs when p is small (see Albert, Jeong, and Barabási 2000). When $p = 1$, the trading network is complete.

to analyze when these institutions mitigate the bargaining friction. We can interpret the inhomogeneous trading network with a small number of trading relationships as an extreme case of a core-periphery financial architecture in which a small number of dealers or banks intermediate trades between a large number of market participants (banks, hedge funds, or firms), and some of these market participants can also trade directly with each other.

My theoretical framework allows us to study trading efficiency in any financial architecture using analytical or numerical computation of the expected welfare loss as I describe in section 5.3. I study a financial architecture with only one large intermediary to keep the analysis simple. In a star-shaped financial architecture, the agent with $n-1$ trading relationships can be a seller, a buyer, or an intermediary, in which case, he needs first to buy an asset from a seller and to sell it to the final buyer. This financial architecture is different from a centralized exchange that can be modeled as a complete network in which all agents can trade directly.

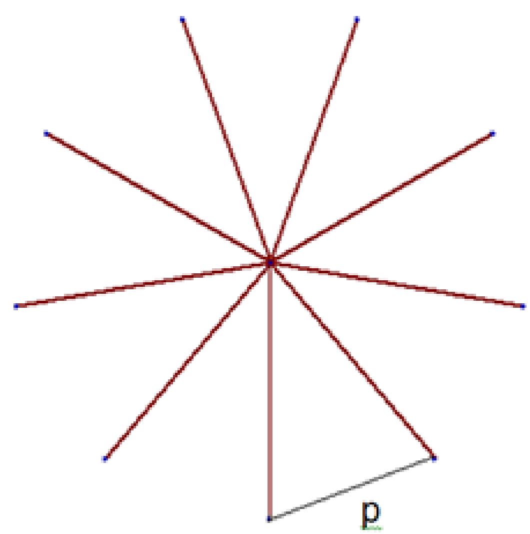

Figure 4: Illustration of an inhomogeneous trading network with 10 traders

Let $W = \{w_{FB}, w_{SB}, w_{TB}\}$ be a vector of probabilities that the agent with $n-1$ trading relationships has the highest, second-highest, or third-highest valuation, respectively. We can interpret these probabilities as a type of a financial institution. If $w_{TB} = 1$ then the agent with $n-1$ trading relationships is a pure intermediary, who only buys and resells assets. If $w_{SB} = 1$ then this agent has a comparative advantage in managing risks relative to most other trading agents, but a more efficient holder of this risk still exists. If $w_{FB} = 1$ then this agent is the efficient buyer of an asset because he is best able to manage the underlying risk. In reality, large interconnected financial institutions both intermediate trades and keep assets on their books, that would correspond to $w_{FB}, w_{SB}, w_{TB} \in (0,1)$.

The probability of inefficient allocation in an inhomogeneous trading network is given by

$$PIA^{ih}(n,p,b,W) = w_{TB}(1-b)(1-p)\left(\frac{1}{n} + \frac{n-3}{n}p(1-p)\right). \tag{5}$$

The first term is the probability that the agent with $n-1$ deterministic trading relationships has the third-highest valuation. If he has the highest or the second-highest valuation then the equilibrium allocation will always be efficient in the simple economy. The second term is the probability that the second-highest valuation is above the bargaining ability parameter b, assuming v is uniformly distributed between zero and one. If v is smaller than b then the agent with the second-highest valuation will always sell the good to the agent with the highest valuation, because b is the resale price of the large intermediary. The third term is the probability that the agent with the second-highest valuation is not a direct trading partner of the agent with the highest valuation. The last term represents the probability that the agent with the second-highest valuation gets the endowment or that one of the $n-3$ agents with the third-highest valuations, excluding the most connected agent, gets the endowment, times the probability that this agent is a direct trading partner of the agent with the second-highest valuation but not of the agent with the highest valuation. To compute the expected welfare loss, we need to account for the loss given inefficient equilibrium allocation, which is $\int_b^1 (1-v)dG(v)$. When v is uniformly distributed between 0 and 1, and each agent is equally likely to receive the endowment, the expected welfare loss in an inhomogeneous trading network is given by

$$EWL^{ih}(n,p,b,W) = w_{TB}(0.5-b+0.5b^2)(1-p)\left(\frac{1}{n} + \frac{n-3}{n}p(1-p)\right). \tag{6}$$

In the inhomogeneous trading network, the probability of inefficient allocation and the expected welfare loss are not monotonic in the degree of network completeness for the same reason they are not monotonic in the homogeneous trading network. With probability $p(1-p)$, the agent with the third-highest valuation can sell directly to the agent with the second-highest valuation but not to the agent with the highest valuation. If the agent with the second-highest valuation cannot sell directly to the agent with the highest valuation, which has probability $1-p$, then the equilibrium allocation will be inefficient for all $v > b$. The probability of inefficient allocation, given the endowment belongs to one of the $n-3$ agents with zero valuation, is proportional to $p(1-p)^2$ and attains a global maximum at $p = \frac{1}{3}$.

In contrast to a homogeneous trading network that allows for several increases and decreases in the expected welfare loss as a function of the degree of network completeness (p), there is one global maximum in the inhomogeneous trading network because the relative number of intermediaries can be at most one (direct trade with the second best and one intermediary to sell to the first best). The degree of network completeness for which the expected welfare loss and the probability of inefficient allocation are maximized is approximately one third because the probability that one of the $n-3$ agents with the third-best valuation gets the good in the initial endowment is close to one in large networks when each agent is equally likely to get the endowment. The above analysis suggests that a policy that decreases a cost to establish or maintain a trading relationship between agents can decrease efficiency relative to a star structure in which a pair of agents requires an intermediary to trade.

The non-monotonic relationship between the degree of network completeness and the expected welfare loss also implies that markets with less intermediation are not necessarily more efficient. For example, in an inhomogeneous trading network, the expected number of intermediaries required to facilitate trade between any pair of agents, excluding the agent in the center, is $1-p$ because with probability p, any pair of agents can trade directly, and with probability $1-p$, they require one intermediary. Therefore, the expected number of intermediaries is decreasing in p, but the expected welfare loss can increase in p. Hence, a financial architecture in which agents require on average fewer intermediaries to trade does not necessarily result in a higher trading efficiency.

Table 2 reports different measures of trading efficiency for inhomogeneous trading networks of different sizes and degrees of completeness. I assume the agent with $n-1$ deterministic trading relationships is a pure intermediary, has private valuation of zero ($w_{TB}=1$), and that the bargaining ability of each seller is half ($b=0.5$).[37] For a proper comparison of homogeneous and inhomogeneous trading networks, I set the probability of a trading relationship in the inhomogeneous trading network to be such that the expected number of trading relations in both networks is the same.[38]

[37] The maximum number of intermediaries (D^{ih}) is one in an inhomogeneous trading network because the agent with $n-1$ trading relationships can facilitate trade between any pair of agents.

[38] In the homogeneous trading network with degree of completeness p^h, the expected number of trading relationships is $p^h\frac{n(n-1)}{2}$. In the inhomogeneous trading network, the expected number of trading relationships is $p^{ih}\frac{(n-1)(n-2)}{2}+n-1$. In Table 2, I set $p^{ih}=p^h\frac{n}{n-2}-\frac{2}{n-2}$, such that the expected number of trading relationships and the expected number of trading partners are the same as in Table 1.

From the comparison of the results in Table 1 and Table 2, we learn the inhomogeneous trading network allows agents to achieve a more efficient allocation on average for small p, but the expected welfare loss is the same as when the maximum number of intermediaries in the homogeneous trading network is one. The intuition for this result is that the benefit of the agent with $n-1$ trading relationships is that any two agents can trade through at most one intermediary. I use equation 4 to find that when $p > \sqrt{\frac{\log(n)}{n}}$, any two agents can trade through at most one intermediary in a homogeneous trading network.[39] When any two agents require at most one intermediary to trade, the benefit of having a large intermediary who can trade with everyone disappears. For example, in an inhomogeneous trading network with 1000 (10000) agents and one large interconnected intermediary, if each of the agents has 83 (304) trading partners on average then removing one large interconnected institution will not affect efficiency because the trading network is sufficiently dense to allow each pair of agents to trade using at most one other agent as an intermediary.[40]

Figure 9 plots the expected welfare loss in a homogeneous (dashed line) and an inhomogeneous trading network (solid line) for the trading network of 1000 agents, which is approximately the number of banks in the federal funds market (Bech and Atalay 2010), with $b = 0.5$ and $w_{TB} = 1$. The threshold $\sqrt{\frac{\log(n)}{n}}$, the degree of network completeness above which both trading networks have the same expected welfare loss, is marked on the plot with a dot. We can see the threshold is a good approximation for p at which the two lines merge. The difference in the expected welfare loss is the highest when the degree of network completeness is the smallest. Based on the DTCC data from August 13, 2010, a non-dealer buys approximately 0.8% to 1.2% of all credit derivative contracts directly from another non-dealer and the rest from one of the dealers.[41] We can think of this number as a proxy for p, which measures the probability that two customers are direct trading partners. For this interval of p, the expected welfare gain from the large interconnected intermediary is approximately 4%.

In summary, when a large interconnected financial intermediary is the first-best or the second-best holder of the traded asset then the equilibrium allocation in a simple economy

[39] It is the first term of $\check{p}(1, n)$ because the second term is small for large trading network. Interestingly, this threshold is the square root of the threshold for the random network to be connected derived by Erdős and Rényi (1961).

[40] In this analysis, I assume the large interconnected financial institution has the same bargaining ability as all other agents. If this institution has higher bargaining ability then it will improve welfare even when markets are very interconnected.

[41] Table 1 in `http://dtcc.com/products/derivserv/data_table_i.php`.

Table 2: **Efficiency of trading in an inhomogeneous trading network**

n	K	D^{ih}	EWL^{ih}	PIA^{ih}	$EWL^{ih}\|q_{SB}=1$
100	5	1	0.5%	1.9%	12.1%
100	10	1	1.	3.9	11.5
1000	10	1	0.1	0.4	12.4
1000	15	1	0.2	0.7	12.3
1000	30	1	0.3	1.4	12.1
1000	100	1	1.	4.	11.3
1000	300	1	1.8	7.4	8.8
10000	20	1	0	0.1	12.5
10000	40	1	0	0.2	12.5
10000	80	1	0.1	0.4	12.4
10000	100	1	0.1	0.5	12.4
10000	200	1	0.2	1.	12.3
10000	3000	1	1.8	7.4	8.8

is always efficient. The institution with the third-highest valuation improves efficiency in a large market if the degree of network completeness is small. First, large interconnected institutions decrease the amount of intermediation in the market and therefore decrease the bargaining friction. Second, in a star-shaped financial architecture, the relative number of intermediaries between each seller and all potential buyers is zero because each pair of agents at the periphery requires the intermediary in the center to trade. This intermediary will always sell to the agent with the highest private valuation. Therefore, a star-shaped financial architecture is more efficient than a financial architecture in which some sellers can sell an asset directly to a final buyer who is not the first-best holder of the asset. The policy implication of this result is that moving from a core-periphery financial architecture to a financial architecture in which all agents have a similar but small number of trading partners can decrease welfare.

The next section examines the costs of a financial architecture with a large interconnected financial institution.

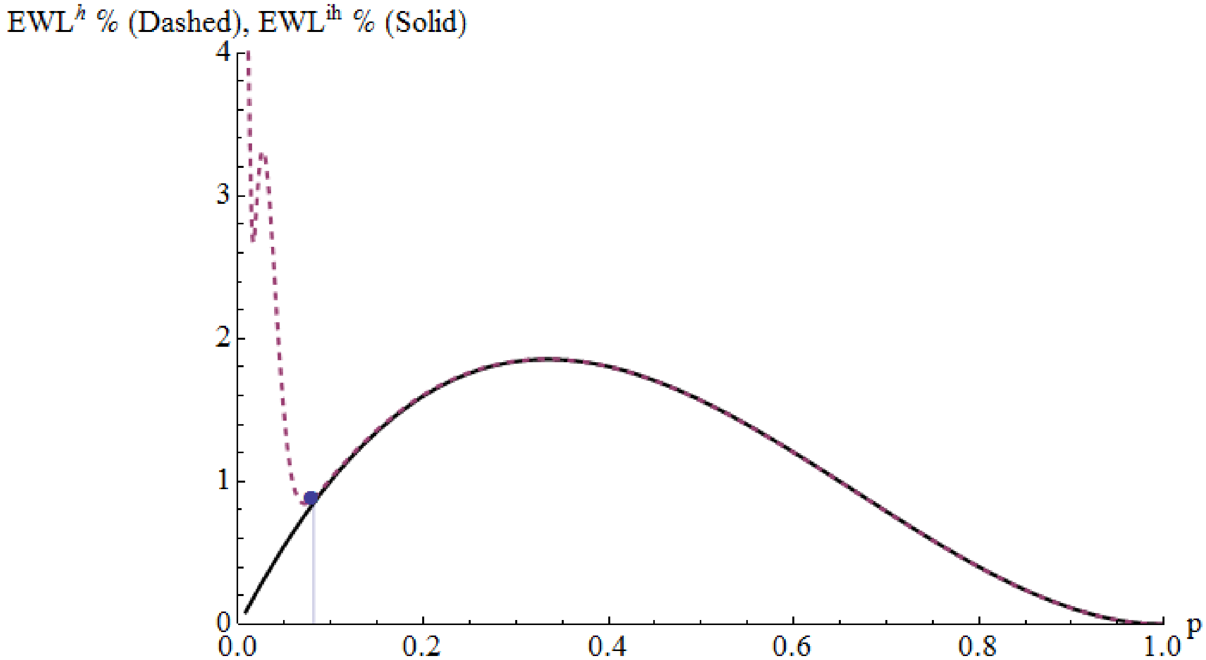

Figure 5: Expected welfare loss with and without a large intermediary

4.1 Moral hazard cost of a large interconnected institution

The previous section presents the benefits of the financial architecture with a large interconnected financial institution. In this section, I study a moral hazard cost in a simple economy that has this financial architecture. If a large interconnected institution expects to be bailed out because of its importance to the financial system then its trading decisions can result in an inefficient allocation of resources.[42] Therefore, we should take into account both the benefits and the costs in the analysis of the desired financial architecture.

A policy that limits the importance of any particular financial institution by changing the financial architecture reduces the moral hazard cost. The government can prefer to bail out a systemically important institution because of the institution's large balance sheet

[42]The chairman of the Federal Reserve System in his testimony to the Financial Crisis Inquiry Commission stated that the existence of too-big-to-fail firms generates a severe moral hazard problem in the long-run (Bernanke 2010, p. 20). I focus on the moral hazard cost of large interconnected institutions in my further analysis to show how my theoretical framework allows a comparison of trading efficiency improvement and the moral hazard cost of large interconnected institutions. A financial architecture with a small number of large interconnected institutions can have additional costs besides the moral hazard cost. For example, an operational risk of these institutions can become a systemic risk to the economy because of their central position.

(too-big-to-fail), large gross position in derivatives (too-interconnected-to-fail), or large costs to establish another such intermediary that will mitigate the bargaining friction in future trades. If all institutions have a small number of trading partners, the above reasons for a bailout disappear. For example, consider a policy that limits banks in the federal funds market to have at most 10 counterparties. If the maximum number of counterparties in this market is reduced from 165 to 10, then each bank is unlikely to be bailed out.[43] However, the policy that limits the number of trading partners of each bank will result in an increase in the number of intermediaries between any pair of banks, and consequently in a higher expected welfare loss due to the bargaining friction.

A tension exists between efficiency of a financial architecture and its stability. When all market participants are restricted to trade with a small number of counterparties, no institution is too essential to the financial system and its failure is unlikely to pose a systemic risk to the economy. However, this financial architecture requires a lot of intermediation, which can decrease efficiency because of the bargaining friction. A financial architecture with one large interconnected financial institutions can improve efficiency, but its failure can result in a failure of other institutions as well. In addition, a financial architecture with one large interconnected institution is unstable in the sense that if the institution in the center of a star-shaped financial architecture fails, the trading network becomes disconnected and trading between market participants will require establishing new trading relationships or a new large interconnected institution.

A simple approach to study the trade-off between efficiency and stability of different financial architectures is to assume large interconnected institutions will always be bailed out ex-post, and to ask how this bailout policy affects efficiency of trading ex-ante. Anticipation of a bailout creates a moral hazard problem that I model as a distortion in the private valuation of the institution that expects to be bailed out. The distortion introduces a moral hazard problem because it affects equilibrium trading decisions. This assumption allows me to compare efficiency of different financial architectures without modeling formally the notion of stability. The trade-off between the benefit and the cost of large interconnected financial institutions is the focus of the further analysis.

I model the effect of the bailout in a reduced form, assuming it increases the private valuation of the agent with $n-1$ trading relationships by $0 < \alpha < 1$. The rationale behind

[43] Bech and Atalay (2010) construct a network of trades between 479 banks that participated in the federal funds market on September 29, 2006. The bank in the center traded directly with 165 other banks, and those banks traded directly with 271 banks that did not trade directly with the bank in the center.

this assumption is as follows. The private valuation of each agent for a financial asset depends on the cash flows in future states. If the cash flow is negative in some future state then without government's bailout, the large financial intermediary, like any other agent, bears the full loss. The large intermediary bears only part of the loss from holding the asset on its balance sheet if it is bailed out by the government in some future states of the world. From the perspective of the large interconnected institution, the ex-post bailout from the government increases the institution's private valuation for the traded asset by some amount, which is the present value of the government's subsidy to prevent the failure of this institution in some future states.[44] I refer to the change in the private valuation due to the ex-post bailout as the *bailout put*. The bailout put can have a positive or negative effect on efficiency, as I discuss below. When the effect is negative, I call it a *moral hazard problem* and call the corresponding welfare loss a *moral hazard cost*.

The effect of the ex-post bailout depends on the private valuation of the large interconnected institution. If this institution has the highest valuation then no welfare loss due to the moral hazard problem occurs in the endowment economy, assuming the probability of the crisis is not affected. If this institution has the second-highest valuation then the moral hazard cost is positive because some assets are inefficiently accumulated on its balance sheet. If this institution has the third-highest valuation then the bailout put improves efficiency because an increase in its private valuation increases the price that this institution gets from the agent with the highest valuation. Hence, he is more likely to buy from the agent with the second-highest valuation.

I compute the change in the expected welfare loss due to the bailout put for each of the cases separately and then I combine the costs and benefits of a large intermediary.

The expected welfare loss due to the bailout put is given by

$$MH^{loss}(\alpha, G, W) = w_{SB} \int_{1-\alpha}^{1} (1 - v) dG(v). \tag{7}$$

With probability w_{SB}, the agent with $n-1$ trading relationships has the second-highest valuation. The welfare loss is $(1-v)$ if his private valuation together with the present value of the government's put is higher than the first-best valuation: $1 - \alpha < v < 1$. When a large interconnected institution's private valuation plus α is above 1, all agents will sell the good to this institution, even if the initial endowment is first best. Therefore, when the

[44] Here I assume that the private valuation for a risky asset is larger because of the ex-post bailout. In the market for insurance the ex-post bailout can decrease the private valuation for insurance because the government provides already part of the insurance.

agent with $n-1$ trading relationships has the second-highest valuation, the welfare loss due to the moral hazard problem depends only on the distribution of v and on the value of α.

The expected welfare gain due to the moral hazard problem is given by

$$MH^{gain}(n,p,b,\alpha,G,W) = w_{TB}(1-p)\left(\frac{1}{n}+\frac{n-3}{n}(1-p)p\right)\int_b^{b+(1-b)\alpha}(1-v)dG(v). \quad (8)$$

With probability w_{TB}, the large intermediary has a third-highest valuation. With probability $1-p$, the agent with the second-highest valuation cannot sell directly to the agent with the highest valuation. He gets the good in the initial allocation with probability $1/n$ or buys it from one of the $n-3$ agents with the third-highest valuation. For this purchase to happen, one of the agents with the third-highest valuation needs to get the endowment and he needs to be able to trade directly with an agent with the second-highest valuation (prob. p) and not be able to trade with the agent with the highest valuation (prob. $1-p$). Without the bailout put, the large intermediary sells the good to the agent with the highest valuation for b. With the bailout put, he sells for $\alpha+b(1-\alpha)=b+(1-b)\alpha$. The decrease in the welfare loss because of the ex-post bailout is $\int_b^{b+(1-b)\alpha}(1-v)dG(v)$. The welfare gain increases in the degree of network completeness as long as $p<1/3$ and $\alpha>0$. When $w_{TB}=1$ and $\alpha=0$, the social benefit of the large institution is positive only if the degree of network completeness is below $\sqrt{\frac{\log(n)}{n}}$, as I discussed before. When $w_{TB}=1$ and $\alpha>0$, the benefit from the large intermediary is positive for $p<1$.

The net effect on efficiency of the bailout put for large trading networks, assuming v is uniformly distributed, is given by[45]

$$MH^{\text{net loss}}(n,p,b,\alpha,W) = w_{SB}\frac{\alpha^2}{2} - w_{TB}(1-p)^2p(1-b)^2\alpha(1-\frac{\alpha}{2}). \quad (9)$$

Figure 6 compares the expected welfare loss in a homogeneous trading network (black line) with the expected welfare loss in an inhomogeneous trading network for two cases: $w_{SB}=1$ (horizontal line), and $w_{TB}=1$ (dashed line), assuming $n=1000$, $b=0.5$, and $\alpha=0.2$. When $w_{SB}=1$, the moral hazard problem results in a welfare loss. When $w_{TB}=1$, the bailout put improves efficiency because it effectively increases the bargaining power of the large interconnected intermediary but does not result in an inefficient allocation in equilibrium. If $0<w_{SB}<1$ then the net benefit of a large interconnected institution is

[45]When n is large, $1/n\approx 0$, $(n-3)/n\approx 1$.

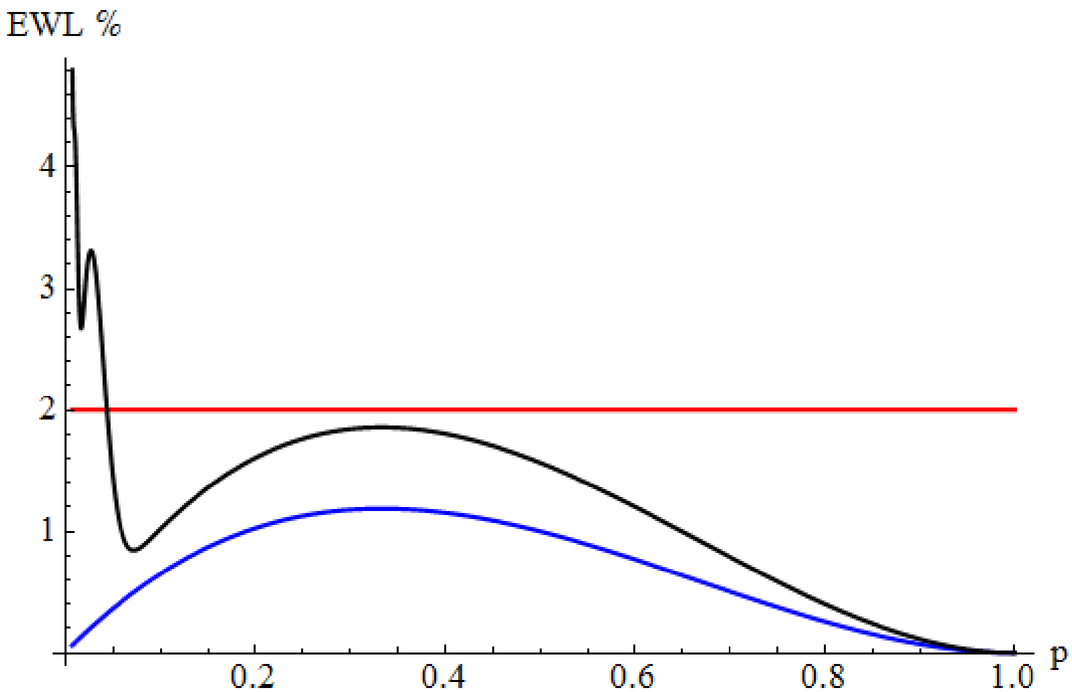

Figure 6: Benefits and costs of a large interconnected institution

a linear combination of the two extreme cases ($w_{TB} = 1$ and $w_{SB} = 1$), such that the net benefit can be positive in several ranges of p.

We learn from Figure 6 that in the reasonable range of $p \approx 1\%$, having a large interconnected financial institution produces a net benefit of 2% even when the value of the bailout put is as large as 20% of the maximum surplus ($\alpha = 0.2$). In general, whether having large interconnected institutions improves efficiency depends on the private valuation of this institution, the value of the bailout put, and the degree of network completeness. Next, I compute indifference curves for w_{TB} given α and for α given w_{SB} such that the net benefits are zero.

In Figure 7 (left), I compute α such that the net benefit of a large interconnected institution is zero, assuming $n = 1000$, $w_{SB} = 1$, and $b = 0.5$. The interpretation of this figure is as follows. If each of the 1000 agents has on average 10 trading partners, and the large interconnected institution has the second-highest valuation, then the net benefit of having this institution is zero when the value of the bailout put is 30%. For this level of α, the improvement in trading efficiency is offset by the welfare loss due to inefficient accumulation of assets by the institution that can trade directly with all market participants. When $\alpha > 0.3$, the financial architecture without the large interconnected institution is more efficient because the moral hazard cost is larger than the benefit from improvement in trading efficiency. For different values of p, the figure provides a value for α such that the net benefit of a having large interconnected institution is zero.

In Figure 7 (right), I compute w_{SB} such that the net benefit of the agent with $n-1$ trading relationships is zero, assuming $n = 1000$, $\alpha = 0.3$, $b = 0.5$, and $w_{FB} = 0$. If α is above the indifference curve in Figure 7 (left plot) then the moral hazard costs of the large interconnected institution outweigh its benefits from mitigating the bargaining friction. Similarly, if the probability that the agent with $n-1$ trading relationships has the second-best valuation is above the indifference curve in Figure 7 (right) then the moral hazard costs outweigh the benefits. The indifference curve accounts for an interesting trade-off that is likely to exist in more general environments. The trade-off is that on one side, we want large interconnected institutions that have high private valuations, because higher private valuations allow them to get better resale values and intermediate more trades. On the other side, the higher their private valuation, the more likely these institutions are to inefficiently keep assets on their balance sheets because of the anticipated bailout.

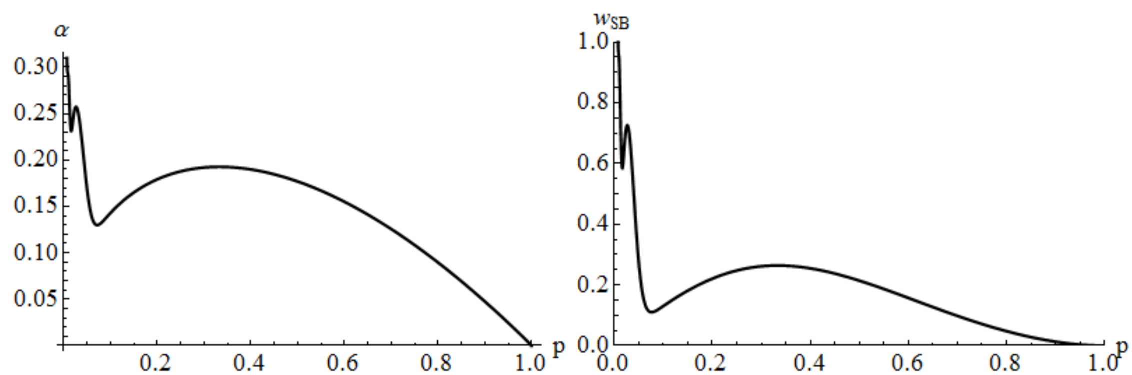

Figure 7: When is a large interconnected institution is too interconnected to exist?

Overall, the above analysis characterizes the conditions when a financial architecture with one large intermediary is beneficial. Many reasons exist for the ex-post bailout of the large interconnected financial institution. However, the decision about the financial architecture going forward is different from the decision of whether to bail out ex-post. The question of whether large financial intermediaries should exist depends on how beneficial they are during times in which they do not require a bailout but they anticipate a bailout in some future states of the world. My analysis shows that in many cases, a large interconnected intermediary improves trading efficiency by more than the moral hazard cost.

The theoretical framework I develop allows for a comparison of financial architectures with more than one large interconnected institutions and with general distribution of valuations and endowments. The reduced-form model of the moral hazard problem allows for a comparison of financial architectures in terms of their trading efficiency for any distortion in private valuations of large interconnected institutions. Comparison of homogeneous and inhomogeneous trading networks combined with the reduced form model of the moral hazard problem allow me to account for both the costs and the benefits of large interconnected institutions in the same analytical framework and to study the relationship between the financial architecture and efficiency of the resource allocation process in OTC markets.

5 Extensions

In this section, I consider two alternative trading mechanisms: bargaining with the outside option to sell the good, and a second-price sealed-bid auction. I also outline the procedure to calculate the expected welfare loss in a general economy.

5.1 Bilateral bargaining with an outside option to sell

Assume the seller can approach another buyer if negotiations break down. Specifically, if i sells the good to j then the outside option of i is either to keep the good or sell it to another trading partner. I assume that if negotiations between i and j break down, i never trades this good with j again. For example, consider a trading network with three agents, where A has a trading relationship with B and B has a trading relationship with C, but A and C cannot trade with each other directly. The outside option of B when he negotiates with C is to keep the good or sell it to A for a price based on B's private valuation as an outside option.

The equilibrium valuation of agent i is given by

$$P_i'(g) = \max\{V_i, \max_{j \in N(i,g)} P_i'(g \backslash ij) + B_i \left(P_j'(g) - P_i'(g \backslash ij) - S_j'(g \backslash ij)\right)\},$$

where $P_i'(g \backslash ij)$ is the valuation of i in a trading network $g \backslash ij$, which is a trading network g without a trading relationship between i and j, and $S_j'(g \backslash ij)$ is the gain from trade that j

receives if negotiations between i and j break down. If i's equilibrium outside option is to keep the good then $S'_j(g\backslash ij) = 0$. If i's equilibrium outside option is to sell the good but j does not buy it in equilibrium then $S'_j(g\backslash ij) = 0$. However, in a trading network in which multiple trading paths from a seller to a buyer are possible, j can buy the good from some other agent if negotiations with i break down, such that $S'_j(g\backslash ij) > 0$.[46]

The definition of prices is recursive because the outside option depends on prices in a subnetwork, which depend on an outside option based on prices in a subsubnetwork and so on. The computation of the equilibrium requires us to solve for valuations in all connected subnetworks of a trading network g, starting with subnetworks of size two. Next, I show that the equilibrium allocation can be inefficient when a seller has the outside option to sell the good.

Assume agent x can trade with agent y and agent z can trade with agent y, but x and z cannot trade with each other directly. Let $B_y = 1 - \epsilon$, $V = \{V_x = 1 - \epsilon^2, V_y = 0, V_z = 1\}$, and $E_x = 1$. The allocation is inefficient if x keeps the good. The outside option of x when he trades with y is his private valuation. The price at which y can sell the good to z depends on y's outside option, which is the price he gets when he sells to x: $P(y, x; g\backslash\{y, z\}) = (1 - \epsilon)(1 - \epsilon^2)$. Hence, the price at which y can sell the good to z is $P(y, z) = \epsilon(1 - \epsilon)(1 - \epsilon^2) + (1 - \epsilon)1 = 1 - \epsilon^2 - \epsilon^3 + \epsilon^4$. Therefore, $V(x) = 1 - \epsilon^2 > P(y, z)$ for any $0 < \epsilon < 1$, and the equilibrium allocation is inefficient. The intuition for this result is that intermediary's ability to sell the good to other trading partners can increase the share of surplus he receives from the final buyer, but as long as he cannot extract the full surplus, efficient allocation is not guranteeed. The inefficiency happens when the owner of the good prefers to keep it because the price the intermediary can pay is not sufficiently high.

5.2 Auctions

In this section, I consider an incomplete information environment. Assume the private valuations of the agents are their private information. The trading protocol is such that the good's owner auctions it to his trading partners, setting his private valuation as the reservation price. Before each agent submits a bid, he asks his trading partners to submit

[46]Stole and Zwiebel (1996) study a model of bargaining over wages between a firm and employees. The outside option of the firm when it negotiates with an employee is defined by the firm's surplus in bargaining with the remaining set of employees, which is similar to bargaining in a subnetwork in my model.

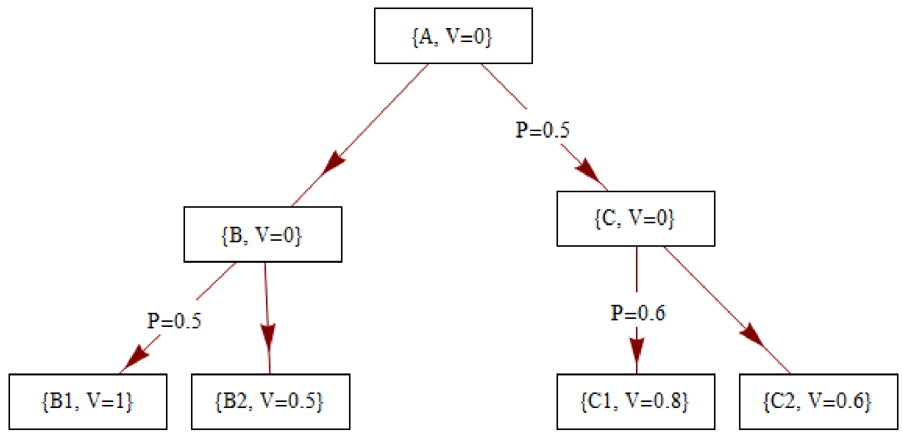

Figure 8: Inefficient Allocation in Sequential Auctions

their bids to him.[47] After all bids are submitted, the good is allocated to the final buyer.

Next, I show that when a seller cannot receive bids from all market participants, each auction is efficient, but a sequence of auctions can result in an inefficient allocation. I provide an example of the inefficient allocation when sellers use a second-price sealed-bid auction as a price-setting mechanism.[48] Consider a case presented in Figure 8, where A auctions a good between B and C. All three have zero private valuation for the good. B asks his potential buyers $B1$ and $B2$ to submit their bids. C does the same for his trading partners, $C1$ and $C2$. Assume valuations of the buyers are $V_{B1} = 1$, $V_{B2} = 0.5$, $V_{C1} = 0.8$, and $V_{C2} = 0.6$. Assuming the trading network consists of six agents, $B1$ has the highest valuation for the good. In the second-price auctions, the dominant bidding strategy is to bid the true valuation for the good derived from private value and resale value. The resale value of B is 0.5 because that is the second-highest bid he gets when he auctions the good. The resale value of C is 0.6 because that is the bid $C2$ submits and the price $C1$ pays. Therefore, the highest bid A gets is 0.6 from agent C, who buys the good for 0.5. The final buyer of the good is agent $C1$ and the equilibrium is inefficient. The agent with the highest

[47] I assume they can commit to their bids. In this case, it is sufficient for each agent to know only his trading partners, and there is no requirement to know the full structure of the trading network.

[48] The revenue-equivalence theorem in Myerson (1981) states that other auctions will not generate a higher surplus to the seller than the second-price sealed-bid auction as long as they are ex-post efficient and a bidder with the lowest valuation gets zero profit from participating in the auction. In addition, this paper shows that in an incomplete information environment "no auction mechanism can guarantee to the seller the full realization of his object's value under all circumstances" (page 72).

valuation, $B1$, would want to pay a higher price to B, but in a second-price sealed-bid auction, the price he pays does not depend on his valuation but on the valuation of the other bidder.

Next, I study a trading mechanism M^a that approximates the second-price auction in a market described by any trading network. According to this trading mechanism, each agent's equilibrium valuation is the maximum between his private valuation and the discounted second-highest valuation among his trading partners. Here I assume that each agent has at least two trading partners, which is not a strong assumption for large trading network with many traders.[49] Formally,

$$M_i^a(P; V, \delta, g) = P_i^a = \max\{V_i, \delta \max_2 P_{j\in N(i,g)}^a\}, \tag{10}$$

where $\max_2 P_{j\in N(i,g)}^a$ is the second-highest valuation among all valuations of i's trading partners in a trading network g. To avoid situations in which agents trade back and forth at a price above the highest private valuation, I assume a seller delivers the good immediately but a buyer transfers a payment with a small delay, such that the seller receives a payment discounted by $0 < \delta < 1$, which can be arbitrary close to one. I make this assumption to avoid trading cycles of the following type. Assume three agents are trading on a complete network and each agent has the same equilibrium valuation above the highest private valuation. Then the second-highest valuation among trading partners of each agent equals the equilibrium valuation, and the good can be traded an infinite number of times at a price above its fundamental value. Discounting prevents this situation because the seller always gets a price smaller then the valuation of the buyer.

Proposition 7. *There are no trading cycles in equilibrium when agents trade using trading mechanism M^a.*

The intuition for this result is similar to the intuition why there are no trading cycles when agents use bilateral bargaining. Prices increase on the equilibrium trading path because intermediaries never loose money. To get a trading cycle, prices need to increase all the time. Agents with finite budget constrain cannot trade with infinite prices, so the only option is that prices increase but converge to a finite limit. I show in the proof that it can not happen.

[49]It is possible to allow bilateral bargaining in case there is only one trading partner, but then we should maintain the assumption that valuations are common knowledge. With the second-price auction, agents can ask for bids their trading partners and prices are set without knowledge of the trading network structure or of the private valuations of all the agents.

Proposition 8. *The vector of equilibrium valuations when agents trade using trading mechanism M^a is unique for any network structure and any vector of private valuations.*

The proof of the proposition uses the contraction mapping theorem. To compute equilibrium valuations, we can use a similar iterative approach as the one I describe in section 7.2 for the bilateral bargaining.

The analysis of the alternative trading protocols suggests the bargaining friction is not specific to the price-setting mechanism used in the main part of the paper. As long as each seller cannot extract the full surplus in each trade, the resource allocation can be inefficient both in the complete information and incomplete information environments.

The contraction mapping theorem I used to prove the uniqueness of equilibrium when agents trade using trading mechanism M^a, can also be used to prove uniqueness of equilibrium under any trading mechanism that is a contraction mapping. A general trading mechanism $M(P)$ defines what is the valuation for the good by each agent given the valuations of his direct trading partners, his private valuation and some exogenous set of parameters that can be a vector of bargaining abilities or discount coefficients of the agents. If we can show that for any two valuation vectors $P, P' \in [0,1]^n$, exists $\rho \in (0,1)$ such that $\max\limits_{i \in N}\{|M_i(P) - M_i(P')|\} \leq \rho \max\limits_{i \in N}\{|P_i - P_i'|\})$ then $M(P)$ is a contraction mapping in a metric space (S, d), where $S = [0,1]^n$ and $d(x,y) = \max_{i \in N} |x_i - y_i|$ is a norm. If $M(P)$ is a contraction mapping then according to the contraction mapping theorem (see Stokey, Lucas, and Prescott (1989), Theorem 3.2), the vector of equilibrium valuation is unique and we can compute for equilibrium using the iterative approach. The ability to solve for the equilibrium trading decisions allows also to compute the expected welfare loss in different economic environments and opens a possibility to compare different trading mechanisms in terms of their efficiency. In the next section, I outline the procedure to compute the expected welfare loss in different economic environments.

5.3 Numerical Solution for the Expected Welfare Loss

In this section, I show how to compute expected welfare loss for general economic environments described by an arbitrary network structure, distribution of private valuations, and any endowment process. This computation allows us to compare different financial architectures in terms of their efficiency. For example, one can compare a financial architecture with a small number of large interconnected financial institutions with the financial architecture where no market participant is too interconnected. This comparison can be used for a normative analysis of consequences of a law that would put restrictions on the size or number of trading partners participants in OTC markets are allowed to have. The numerical procedure can be also used for a positive analysis of OTC markets. A market structure reconstructed based on trades or contracts between market participants has a list of measurable properties. For example, Bech and Atalay (2010) construct a network of trades between banks in the federal funds market and report the distribution of the number of trading partners in this market, the maximum number of intermediaries, the clustering coefficient (which measures how likely two trading partners of a bank to trade with each other), and so on. Comparing the expected welfare loss in a real market structure with a hypothetical market structure that is similar in some characteristics but different in others can help understand why real markets have certain characteristics.

The numerical procedure for computing the expected welfare loss relies on a solution algorithm (such as one described in section 7.2 for the case of bilateral bargaining) that allows us to solve for equilibrium valuations and trading decisions for a given trading network (g) and vector of private valuations (V). Given the trading decisions of the agents, we can compute the equilibrium allocation for any initial endowment. For example, if agent a receives the endowment and he sells to agent b in equilibrium, and agent b sells to agent c who keeps the good, then the welfare loss is $WL(E_a = 1; g, V) = \max_{i \in N} V_i - V_c$, which means that when agent a gets the endowment, the welfare loss is the difference between the highest private valuation in the market and the valuation in the equilibrium allocation for this endowment.[50]

The next step is to account for the fact that the same bank, hedge fund, or firm does not always get the endowment. Therefore, to compare different financial architectures in terms of efficiency, we want to account for the fact that the endowment is random. In the previous step, we computed the welfare loss for each initial allocation. Then the expected welfare loss when agent i has q_i probability to get the endowment is given by

$$\widehat{EWL}(g, V) = \sum_{i=1}^{n} WL(E_i = 1; g, V) q_i. \tag{11}$$

In other words, the expected welfare loss is the welfare loss given that agent i gets the

[50]Here I assume the trading network is connected such that the allocation with the highest private valuation in the market is feasible. However, we can also apply this approach to trading networks that are not connected. In this case, the welfare loss is the difference between the highest valuation among all feasible allocations for this endowment and the valuation in the equilibrium allocation.

endowment ($WL(E_i; g, V)$), multiplied by the probability that agent i gets the endowment (q_i), summing over all agents.

We can extend the analysis further by allowing valuations of the agents not to be constant. The idea is that for some assets, one agent has the highest valuation, but for other assets, another agent might have the highest valuation. Therefore, to compare efficiency of different financial architectures we want to account for the randomness in valuations, which we can do by drawing a vector of valuations from a distribution with CDF $G(v)$, computing the expected welfare loss (equation 11) for each draw, and then taking the average across the draws. In this computation, the trading network is the same but valuations of the agents change with each draw. Another possibility is to keep the valuations vector fixed but to assign these valuations to the agents randomly. For example, only three types of valuations might be present in the market, as in the simple economy (see definition 3): some agent has the highest valuation, some agent has the second-highest valuation, and other agents have zero valuation for an asset. For example, banks in the federal funds market that satisfied their reserve requirements are willing to pay zero interest rate on a loan in the federal funds market if there is no interest payment on these reserves. However, even when valuations are fixed, who has these valuations can change. In other words, what matters for efficiency is not only the vector of valuations but also the position of the agents with these valuation in the trading network. If the agent with the highest valuation can always trade directly with the agent with the second highest valuation then the equilibrium allocation will always be efficient (see discussion in section 3.1). In a homogeneous trading network in which every pair of agents have a trading relationship with probability p, the agent with the highest valuation can trade directly with the agent with the second-highest valuation only with probability p, such that efficiency of trading is not guaranteed.

The numerical computation of the expected welfare loss can also be used for a simple economy and homogeneous trading network, which I study in section 3.2. In the simple economy, one agent has the highest valuation and one agent has the second-highest valuation. Computing the expected welfare loss requires the following steps. First, we draw a trading network in which the probability of each trading relationship is p. For each trading network, we can compute the expected welfare loss (equation 11) and then compute the average expected welfare loss across different trading networks. If we draw J trading networks then the average welfare loss is given by

$$\widehat{EWL}^s(V^s) = \frac{1}{J}\sum_{j=1}^{n} \widehat{EWL}(g_j^h, V^s), \tag{12}$$

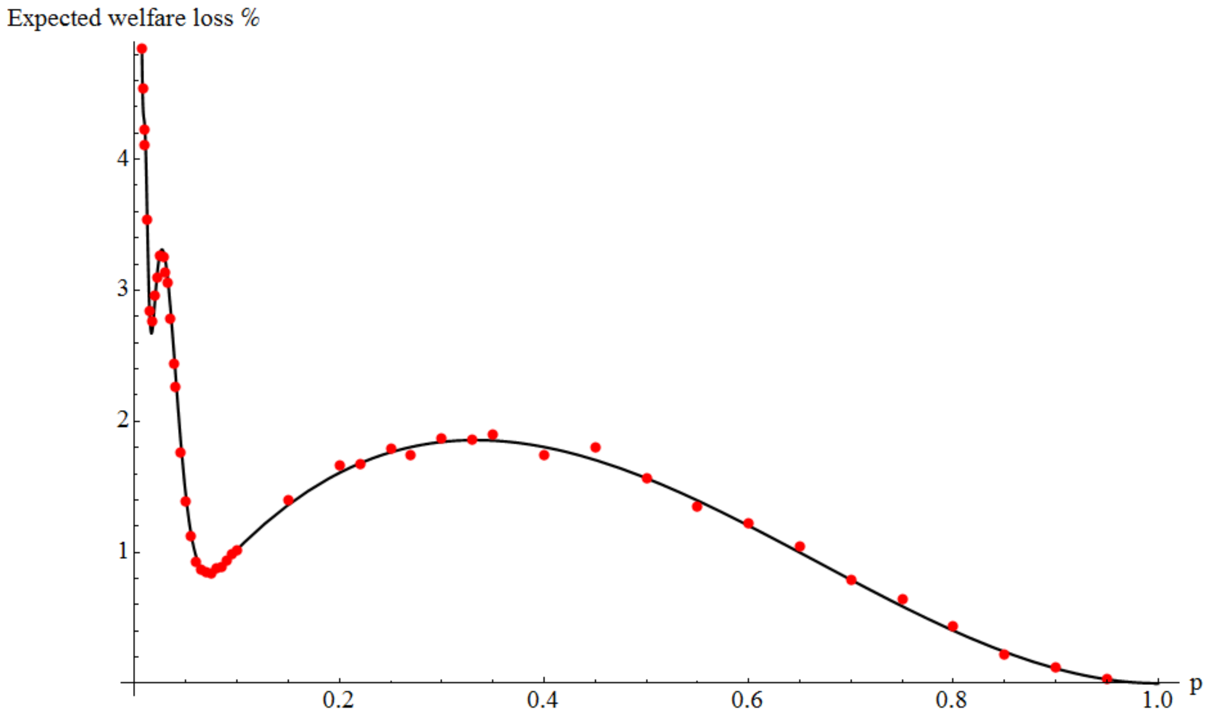

Figure 9: $n = 1000$, $b = 0.5$, $v \sim U[0,1]$, uniform endowment. Red dots represent the results of the numerical solution for the expected welfare loss.

where $\widehat{EWL}(g_j^h, V^s)$ is the expected welfare loss estimate in a homogeneous trading network g_j^h when the vector of private valuations (V^s) satisfies the restrictions of the simple economy (one agent with valuation 1, one agent with valuation $0 < v < 1$, and $n-2$ agents with a private valuation of 0). The final step in this computation is to allow for randomness in v. The idea is that for some assets, the second-highest valuation is close to the highest valuation, in which case the welfare loss in the event of an inefficient allocation is not substantial. However, for some assets, the difference between the highest and the second-highest valuation can be large. If v is distributed according to a commutative distribution function $G(v)$ then we need to compute the expected welfare loss for different values of v as well. In section 7.3, I show that the probability of an inefficient allocation in the simple economy and homogeneous trading network is the same for all $b^j < v \leq b^{j-1}$. Therefore, we can compute numerically the probability that the equilibrium allocation is inefficient for some v in each range ($1 < v \leq b$ is the first range, $b < v \leq b^2$ is the second range, and so on, whereas $b^{D^h-1} < v < b^{D^h}$ is the last range, where D^h is the maximum number of intermediaries in a homogeneous trading network) and then multiply that probability by the expected welfare loss in this range $\int_{b^{l+1}}^{b^l}(1-v)dG(v)$ (it is the last term in equation 20).

Finally, we need to sum the expected welfare loss in each range. The numerical solution for the expected welfare loss in a homogeneous trading network can be compared with the results obtained using equation 20.

In Figure 9, I present results of the numerical solution for the expected welfare loss in homogeneous trading networks with different degrees of network completeness (red dots) and the analytical solution (black line) when $n = 1000$, $J = 1000$, $b = 0.5$, $v \sim U[0, 1]$, and each agent is equally likely to get the endowment. The comparison shows that the analytical solution is close to the numerical solution, and the difference between the two solutions is at most 0.15%.

6 Conclusion

In this paper, I introduce a bargaining friction that affects trading efficiency in OTC markets. The bargaining friction occurs because each intermediary cannot get the full surplus when he sells a good. As a result, a sequence of bilateral trades can result in inefficient allocation even when each seller trades at the highest available price.

I develop a network-based model of the OTC market to address three questions: When is bilateral bargaining an efficient allocation mechanism? What is the welfare loss due to the bargaining friction and how is it related to the market structure, represented by a network of trading relationships? What are the costs and benefits from the presence of large interconnected institutions in OTC market?

In the model, bilateral prices and trading decisions are determined jointly. I show that in equilibrium, there are no bubbles. I develop an algorithm to solve for equilibrium prices and trading decisions for any structure of the trading network and show that the equilibrium valuations are unique.

The welfare analysis suggests bilateral bargaining is an efficient allocation mechanism only when the trading network is complete: every seller can trade with every buyer directly. I compute analytically the expected welfare loss due the bargaining friction in an incomplete trading network, and characterize when adding more trading relationships decreases welfare. I find the decrease in welfare can be substantial. For example, in a trading network of 10,000 agents, adding 14 million new trading relationships can increase the expected welfare loss by more than 400%.

I apply the theoretical framework to analyze the costs and benefits of a large interconnected financial institution. I provide conditions on the structure of the trading network when the institution improves efficiency. A government can prefer to bail out the large interconnected financial institution ex-post because of its large balance sheet (too-big-to-fail problem), its large gross position in derivative (too-interconnected-to-fail problem), or the cost of establishing another such intermediary that will mitigate the bargaining friction in future trades. I find that when a large interconnected institution is a pure intermediary, an implicit guarantee from the government—a bailout put—improves ex-ante efficiency. However, when the bailout put increases the private valuation of the large intermediary above the asset's value in the first-best allocation, the intermediary will inefficiently accumulate assets on its balance sheet. Overall, my analysis of the costs and benefits of large financial institutions contributes to the debate about the desirable architecture of the financial system.

Besides OTC markets, many other economic environments exist in which agents do not trade on a centralized exchange. The bargaining friction can affect institutional or contractual arrangements in these environments. To understand whether contracts between firms in supplier-customer networks or authority in hierarchical organizations mitigate the bargaining friction, we need to solve for equilibrium allocation with and without these arrangements. The analysis presented in this paper is a first step in this direction.

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7 Appendix

7.1 Solution Algorithm - Recursive Backward Induction

In this section, I describe an algorithm to compute equilibrium trading decisions and valuations.

The algorithm iterates over an ordered set of agents $N = \{1, \ldots, n\}$. The agents are ordered in N from the lowest private valuation to the highest private valuation, such that agent n has the highest private valuation and agent 1 the lowest. The goal of an iteration step t is to compute the optimal trading decision of the agents starting from n and solving recursively until we solve for the equilibrium trading decision of the agent with the lowest valuation.

The algorithm solves for the optimal agent with the t-highest valuation in step t. In $t = 1$, the algorithm solves for the optimal trading decision of the agent with the highest valuation. In step $t = 2$, the algorithm solves for the optimal trading decision of the agent with the second-highest valuation, and so on.

Let Q_{t-1} be the set of agents who keep the good in equilibrium based on the previous iteration steps.

Step 1: $t = 1$, agent n's optimal trading decision is to keep the good because the price cannot be higher than his valuation. Therefore, his valuation for the good is equal to his private valuation: $P_i = V_i$. Assign n to the set of agents who keep the good in equilibrium: $Q_1 = \{n\}$.

Step 2: For $1 < t \leq n$, the computation for the optimal trading decision of the agent with t's highest valuation includes the following steps.

Step 2.1: For all $q \in Q_{t-1}$, we need to consider all possible trading paths between trader t and q.[51]

Step 2.2: For each trading path, solve backwards for prices. Let $H(t, q)$ represent the highest price an agent t can get among all possible trading paths that connect him to agent

[51] A trading path between agents i and j, $T(i, j, g)$, is a sequence of bilateral trades $i_1 i_2, i_2 i_3, ..., i_{J-1} i_J$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, ..., J-1\}$, with $i_1 = i$ and $i_J = j$, such that each agent in the sequence $i_1, ..., i_J$ is distinct and each $i_k i_{k+1}$ pair is distinct. The number of trading paths between any two traders is finite in any finite network because each trading path includes only one trade per pair of traders.

q. Let $H_t^* = \max_{q \in Q_{t-1}} H(t,q)$ be the highest price an agent with t's highest valuation can get if he sells. Then $P_t = \max\{V_t, H_t^*\}$ is the valuation of agent t. Later, I prove the price is consistent with the equilibrium trading decisions of the intermediaries.

Step 2.3: If $V_t > H_t^*$ then assign t to the set of agents who keep the good: $Q_t = Q_{t-1} \cup t$.

Step 2.4: $t = t+1$. Return to step 2 until we solve for the valuation of the agent with the lowest private value.

The equilibrium trading of agent i is $\sigma_i = \arg\max_{j \in N(i,g) \cup i} P_j$. If more than one buyers buy for the same price then seller i picks one of the buyers with the highest valuation randomly. If the private valuation is the same as the highest price then i always sells the good.

Next I prove that equilibrium trading decisions computed based on the algorithm are consistent with equilibrium trading decisions of the intermediaries.

Without loss of generality, assume i's optimal trading decision is to sell, according to the algorithm, to $q \in Q(i-1)$ directly or through intermediaries. If he sells directly to q then it is an equilibrium trading decision of i. If it is an indirect trade in which a and b act as intermediaries then we need to check whether the intermediaries's equilibrium trading decisions are consistent with the trading path that generates the highest price to i. Assume that the trading path is i sells to a, a sells to b, and b sells to q. The proof does not rely on the fact that there are two intermediaries and it holds for any number of intermediaries.

Case 1: Without loss of generality, assume a's optimal trading decision is to sell to $x \neq b$. Let y be the final buyer from x and assume x sells directly to y. The same argument holds even if x sells to y through intermediaries. If $y \in Q(i-1)$ then it contradicts the assumption that selling to q generates the highest price for i. The algorithms checks all trading paths to all agents in $Q(i-1)$, so the algorithm considered a trading path where i sells to a, a sells to x, and x sell to y, potentially through other intermediaries, and rejected the path as suboptimal. Therefore, y cannot be in $Q(i-1)$. If $V_y > V_i$ and y's optimal trading decision is to keep the good then y has to be in $Q(i-1)$, which is a contradiction. If $V_y < V_i < P(i,a) < P(a,x) < P(x,y)$ then the price y needs to pay is above his private valuation, which is a contradiction to the assumption that keeping the good is an optimal trading decision of y. Therefore, a cannot sell to any agent other than b.

Case 2: Without loss of generality, assume a's optimal trading decision is to keep the good. If $V_a < V_i < P(i,a) < P(a,b)$ then a will not keep the good if can sell it to b for a price above V_i. If $V_a > V_i$ and $a \in Q(i-1)$ then i could sell to a for a higher price than to q, but the algorithm considered and rejected this alternative. If $V_a > V_i$ and $a \notin Q(i-1)$ then keeping the good is not an optimal trading decision of a, because all agents with valuations higher than V_i whose optimal trading decision is to keep the good belong to the set $Q(i-1)$. Therefore, a's optimal strategy cannot be to keep the good.

Case 3: So far, I have considered cases in which the optimal trading decision of only one intermediary is different from the one that maximizes price to i. Assume the optimal trading decision of more than one intermediary is different. Based on the previous two cases, we know the last intermediary on the trading path from i to q cannot be part of the group of agents who have a different trading decision, because the optimality of his trading decision to sell to q did not depend on the other agents' trading decisions. Similarly, one intermediary before the last intermediary could not have a different trading decision. By checking trading decisions of all intermediaries starting from the the last and recursively moving to the to the first, we can see neither intermediary deviates.

Therefore, the equilibrium trading decisions of all intermediaries are consistent with the set of trading decisions that generate the highest price to i. If i decides to sell according to the algorithm then the price he sells for is the equilibrium price. Thus the proof is complete.

7.2 Solution algorithm - Contraction Mapping

In this section, I show that the trading mechanism in which prices are set by bilateral bargaining (equation 2.3) is a contraction mapping. I refer to this trading mechanism as $M^b(P; V, B, g)$. If M^b is a contraction mapping then according to the contraction mapping theorem (see Stokey, Lucas, and Prescott (1989), Theorem 3.2), the vector of equilibrium valuation is unique, which I prove in proposition 2, relying on the solution algorithm based on recursive backward induction. The benefit of proving that the bilateral bargaining is a contraction mapping and relying on the contraction mapping theorem is that it allows me to solve for equilibrium valuations and trading decisions in large trading networks using an iterative approach, which I describe later.

The trading mechanism M^b determines each agent's valuation for a good in a trading network g given valuations of his trading partners, his bargaining ability, and his private valuation:

$$M_i^b(P) = P_i = \max\{V_i, \max_{j \in N(i,g)} V_i + B_i(P_j - V_i)\}. \quad (13)$$

The interpretation of the above equation is that each agent's valuation is the maximum between his private valuation and the highest price he can get if he decides to sell to one of his direct trading partners.

Proposition 9. $M^b(P)$ *is a contraction mapping.*

Next, I use proposition 9 and the contraction mapping theorem to define an iterative approach to solve for equilibrium valuations and trading decision.

Step 1: Let $i = 0$ and $P(i) \in [0,1]^n$ be some vector of valuations.

Step 2: Let $i = i + 1$; compute $M^b(P(i-1))$ to get $P(i)$. Specifically, compute each agent's new valuation according to equation (13), assuming valuations of his trading partners are given by $P(i-1)$. After we compute each agent's new valuation we get a new vector of valuations $P(i)$.

Step 3: Check whether $P(i) = P(i-1)$. If the two vectors of valuations are the same, $P(i)$ is the equilibrium vector of valuations. Otherwise, we need to make another iteration by returning to Step 2 and computing $P(i+1)$ until we find the fixed point, such that an additional iteration does not change the vector of valuations. The contraction mapping theorem ensures that this fixed point is unique and can be reached using a sequence of iterations.

Step 4: After we find the vector of equilibrium valuations P^*, we can compute the equilibrium trading decisions of the agents using equation (2.3).

7.3 Computation of the expected welfare loss in a simple economy and homogeneous trading network

In a general economy with heterogeneous private valuations, the expected welfare loss is given by equation 1. In this section, I derive the expected welfare loss in a simple economy with three types of private valuations. The loss vector in the simple economy is given by $L^h = \{1, 1-v, 0\}$. In Figure 10, I present the types of allocations and parameters for transition probabilities between these allocations in the simple economy. If the initial allocation is first-best then this allocation is the equilibrium allocation. If the initial allocation is third-best then eventually the good will be transfered to the second-best allocation with probability $y(v, b, g)$ or to the first-best allocation with probability $1 - y$. If the good is transfered to the second-best allocation then with probability $x(v, b, g)$, it will remain there, and with probability $1 - x$, it will be transfered to the first-best allocation. If the initial allocation is third best then with probability $1 - y + y(1 - x) = 1 - xy$, the equilibrium allocation will be first best and with probability xy it will be second best.

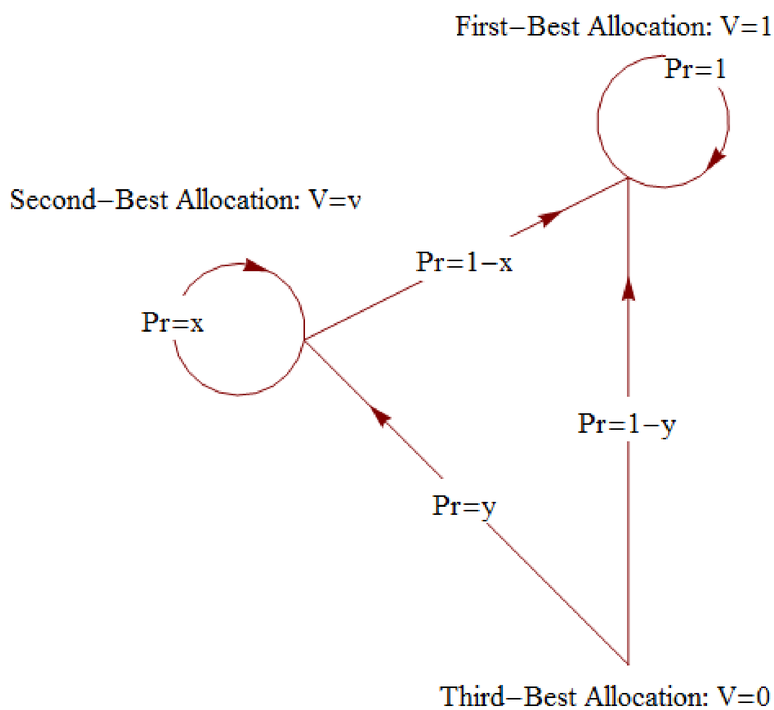

Figure 10: Transitions Between Allocations

Let M^h be a matrix that summarizes the transition probability from any type of initial allocation to any type of equilibrium allocation in a simple economy:

$$M^h = \begin{pmatrix} 0 & xy & 1 - xy \\ 0 & x & 1 - x \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $Q^h = \{q_{TB}, q_{SB}, q_{FB}\}$ be the vector of probabilities that the initial allocation is first best, second best, or third best. For example, if each agent is equally likely to have the good in the initial allocation then $q_{TB}(n) = \frac{n-2}{n}$ and $q_{SB}(n) = q_{FB}(n) = \frac{1}{n}$. Then in the simple economy, equation 1 can be simplified to

$$EWL^h(v, b, g) = Q^h M^h L^h = (1 - v)x(q_{SB} + q_{TB}y). \tag{14}$$

Based on the proof of Proposition 5, in a homogeneous trading network, x is given by

$$x(v, b, n, p) = F(\hat{d} + 1|n, p), \tag{15}$$

where $F(d|n, K)$ is the probability that two randomly picked agents in a homogeneous trading network need at least d intermediaries to trade. The lower bound for $F(d|n, K)$ is derived in section 7.5 of the Appendix.

Next, I compute y, which is the probability that an agent with a valuation of zero, agent TB, sells the good to the agent with the second-highest valuation, agent SB, and not to the agent with the highest valuation, agent FB. Agent TB can sell the good to agent SB for $vb^{d_{sb}}$, where d_{sb} is the minimum number of trades required for this transaction. He can also sell to agent FB for $b^{d_{fb}}$, where d_{fb} is the minimum number of trades required in this case. Therefore, agent TB sells the good to agent SB if $vb^{d_{sb}} > b^{d_{fb}}$ or $d_{fb} > d_{sb} + \hat{d}$. Hence the probability that agent TB sells the good to agent SB directly, or using intermediaries, is given by

$$y(v, b, n, p) = \sum_{j=1}^{\infty} f(j|n, p)F(j + \hat{d}(v, b)|n, p), \tag{16}$$

where $f(j|n, p) = F(j - 1|n, K) - F(j|n, p)$ is the probability that j trades are required for TB to sell to SB (more than $j - 1$ but not more than j). The second term, $F(j + \hat{d})$, is the probability that he needs at least $j + \hat{d}$ intermediaries to trade with the agent with the highest valuation. The intuition for this formula is that for any number of trades required for TB to sell to the SB ($j = 1, 2, 3, \dots$), the number of trades required for TB to sell the good to FB should be $\hat{d}$ trades higher.

Using the formulas for x and y, equation 14 results in the following expression for the expected welfare loss in the homogeneous trading network:

$$EWL^h(v, b, n, p) = (1 - v)F(\hat{d} + 1|n, p)\left(q_{SB} + q_{TB}\sum_{j=1}^{\infty} f(j|n, p)F(j + \hat{d}|n, p)\right). \tag{17}$$

Further, I assume the second-best valuation v is distributed between the valuation in the third-best allocation and the valuation in the first-best allocation according to a cumulative distribution function $G(v)$. In this case, the expected welfare loss becomes:

$$EWL^h(b,n,p) = \int_0^1 EWL^h(v,b,n,p)dG(v). \tag{18}$$

The probability of an inefficient equilibrium allocation depends on v and b, but the relationship is not strictly monotonic because $\hat{d} = \left\lfloor \frac{\log(v)}{\log(b)} \right\rfloor$ is a floor function. For example, $\hat{d} = 0$ for any $v \in (b, 1]$. In general, $\hat{d} = j - 1$ if $v \in (b^j, b^{j-1}]$ for $D^h > j > 1$, where D^h is the largest number of intermediaries between any pair of agents in the homogeneous trading network. The lowest v we need to consider when computing the expected welfare loss is b^{D^h}, because this is the lowest price SB can face.

Lemma 1. *The expected maximum number of intermediaries in the homogeneous trading network is at least*

$$D^h(n,p) = \left\lfloor \frac{\log\left(\frac{\log(n-1)}{-\log(1-p)}\right)}{\log(p(n-1))} + 1 \right\rfloor. \tag{19}$$

To compute x and y, it is sufficient to consider cases where $v = b^j$, in which case $\hat{d} = j$, and then use the computed value of x and y for all values of $b^j < v \leq b^{j-1}$ in the corresponding intervals. This approach allows me to simplify equation 18 to

$$EWL^h(b,n,p,Q^h,G) = \sum_{l=0}^{D^h-1} F(l{+}1|n,p)\left(q_{SB} + q_{TB}\sum_{j=1}^{\infty} f(j|n,p)F(j+l)\right)\int_{b^{l+1}}^{b^l}(1{-}v)dG(v). \tag{20}$$

If $b = 0.5$, v is uniformly distributed between 0 and 1, and each agent is equally likely to get the good in the initial endowment, equation 20 becomes[52]

$$EWL^h(n,p) = \sum_{l=0}^{D^h-1} F(l+1|n,p)\left(\frac{1}{n} + \frac{n-2}{n}\sum_{j=1}^{D^h+2-l} f(j|n,p)F(j+l)\right)\left[v - \frac{v^2}{2}\right]_{0.5^{l+1}}^{0.5^l}. \tag{21}$$

The expected welfare loss accounts for the uncertainty related to the second-highest valuation, the endowment, and the amount of intermediation in the homogeneous trading network.

[52] If there are at most D intermediaries in the trading network then it is sufficient to consider cases where $j \leq D(n,p) + 1 - \hat{d}(v,b)$ because theoretically $F(D+1) = 0$. However, in random network $D(n,K)$ is not deterministic, therefore when I compute $y(v,b,n,p)$ I let j to be at most $D(n,p) + 2 - \hat{d}(v,b)$.

The probability of inefficient allocation in this case is given by:

$$PIA^h(n,p) = \sum_{l=0}^{D^h-1} F(l+1|n,p) \left(\frac{1}{n} + \frac{n-2}{n} \sum_{j=1}^{D^h+2-l} f(j|n,p) F(j+l) \right) \left(0.5^l - 0.5^{l+1}\right). \quad (22)$$

The first two terms are the same as in equation 21. The last term is the probability that $b^{l+1} < v < b^l$.

7.4 Proofs

Proof of Proposition 1. The proof is by contradiction. Assume there is a trading cycle in equilibrium, such that a group of $m \subseteq N$ agents trades between themselves in a cycle. Each agent in this group sells the good an infinite number of times. I distinguish between an equilibrium in which the price in a trade between each pair of agents is the same and the case in which the price can be different. In the first equilibrium, we can focus on a single cycle in which some agent i sells the good to agent $i+1$, and agent $i+1$ sells to agent $i+2$ and so on till the good was sold again to agent i. First, trades happen only if the surplus is positive:

$$P(i+1, i+2) > V_i \text{ for all } i \in M. \quad (23)$$

Second, if each agent decides to sell rather than keep the good then following equation determines the price:

$$P(i, i+1) = (1 - B_i)V_i + B_i P(i+1, i+2) \text{ for all } i \in M. \quad (24)$$

From equations (23) and (24) and from the fact that $B_i \in (0,1)$ for all i, it follows that

$$P(j+1, j+2) > P(j, j+1) \text{ for all } j \in M. \quad (25)$$

Given that inequality (25) holds for all trades in the cycle, it follows that $P(i, i+1) > P(i, i+1)$, which is a contradiction.

In the second case, prices are increasing with each transaction. However, with a finite budget constraint, agents will not be able to transact at infinite prices. Therefore, we can consider only a case in which an infinite strictly increasing sequence of prices $\{P_n\}_{n=1}^{\infty}$ converges to a finite price P ($\lim_{n\to\infty} P_n = P$). Let i be an agent with the highest private valuation among all agents who trade in a cycle. Let $e = P - V_i$ be the difference between the highest valuation and the limit of the infinite price sequence. Agent i sells the good in equilibrium; therefore, $e > 0$. The infinite sequence of prices converges to P; therefore, for any $\epsilon > 0$ there exists N such that $P - P_j < \epsilon$ for all $j > N$. Let P_n^i be the price for which agent i sells the good and $n > N$. Then $P - P_n^i < \epsilon$. Agent i sells to agent j who resells the good for P_{n+1}^j, such that $P - P_{n+1}^j < \epsilon$. An increase in price cannot be larger than ϵ; otherwise, the higher price will be above P, so $P_{n+1}^j - P_n^i < \epsilon$. The price for which agent i sells the good takes into account the resale value of the buyer j: $P_n^i = V_i + B_i(P_{n+1}^j - V_i)$. Next, I use the formula for P_n^i to compute the difference in prices $(P_{n+1}^j - P_n^i)$, which depends on the private valuation of the seller: $P_{n+1}^j - P_n^i = (1 - B_i)(P_{n+1}^j - V_i) < \epsilon$. We can divide both sides of the inequality by $(1 - B_i) > 0$ to get $P_{n+1}^j - V_i < \epsilon/(1 - B_i)$. Then $e = P - V_i = (P - P_{n+1}^j) + (P_{n+1}^j - V_i) < \epsilon(1 + \frac{1}{1-B_i}) = \epsilon\frac{2-B_i}{1-B_i}$. The inequality holds for any $\epsilon > 0$, so it holds for $\epsilon = e\frac{1-B_i}{2-B_i} > 0$, but then we get a contradiction $e < e\frac{1-B_i}{2-B_i} \times \frac{2-B_i}{1-B_i} = e$. Therefore, do not exist trading cycles with an infinite increasing sequence of prices that converges to a finite price.

If there are no trading cycles in equilibrium then there is a non-empty set of agents whoes optimal trading decision is to keep the good in equilibrium. Each trading path should eventually reach one agent in this set. □

Proof of Proposition 2. If neither agent is indifferent between two or more trading decisions then the solution algorithm in section 7.1 generates a unique solution for valuations and trading decisions. Denote this equilibrium solution: EQ.

Assume by contradiction that another equilibrium exists: EQ'. Let $C(g) \subseteq N$ denote a set of agents whose equilibrium valuation and trading decision in EQ' differ from those in EQ.

Let i be the agent with the highest private valuation among all agents in $C(g)$. It cannot be the agent with the highest private valuation because his optimal trading decision is to keep the good.

Case 1: Assume i keeps the good in equilibrium EQ. Then he sells it in equilibrium EQ'. But according to the algorithm, the highest price he could get is below his private valuation. The optimal trading decision of all agents with valuations higher than V_i are the same in both equilibriums. Therefore at least one of the intermediaries in equilibrium EQ' buys the good at a higher price than he can sell it for, which is a contradiction.

Case 2: Assume i sells in equilibrium EQ to a different agent than in EQ'. In this case, some intermediaries between i and the final buyer in EQ should have an suboptimal trading decision in EQ', because as I show in the proof that appears at the end of section 7.1, the highest price to i in EQ is also an optimal trading decision to all intermediaries on this equilibrium path. Therefore i can sell to two different agents in equilibrium, only if i is indifferent. In this case i will have the same equilibrium valuation for the good but different trading decisions. If private valuations are drawn from a continuous distribution then the probability that two or more trading partners have the same valuation is zero. Therefore equilibrium valuations are unique and equilibrium trading decisions are generically generically unique. □

Proof of Proposition 3. First, I prove that if the trading network is complete then the equilibrium allocation is always efficient. Let $M \subseteq N$ be a set of agents with the highest valuations of the good, such that $M = \{i \in N \mid V_i \geq V_j \ \forall j \in N\}$. In equilibrium, trades occur only if the surplus is positive. Therefore $\sigma_i = i$ for all $i \in M$ and $P_i = V_i$ because for each $i \in M$, the highest price i can sell for is lower than his private valuation. Next I prove that any equilibrium with $E_j = 1$ where $j \in N \backslash M$ will be efficient. Assume it is not, such that in equilibrium, $j \in N \backslash M$ keeps the good. The surplus from trade between j and any $i \in M$ is positive and this trade is feasible because everyone can trade directly in a complete trading network. Therefore the price that j receives in this trade is higher than his private valuation. Thus keeping the good is not an optimal trading decision. We got a contradiction. Therefore, in equilibrium, only agents with the highest valuation will keep the good for any initial endowment, valuations, and bargaining ability vectors.

Next I show that for any incomplete network and a vector of bargaining abilities B, one can find a vector of valuations and the endowment, such that the equilibrium allocation is inefficient. If a trading network g is incomplete then it has at least one incomplete subnetwork with three agents, because if any pair of agents could trade directly then the trading network would be complete. Without loss of generality, assume agent X cannot trade directly with agent Z. The trading network is connected, therefore there is at least one agent who can trade directly both with X and with Z. Let agent Y be the agent with the highest bargaining ability among all agents who can trade directly with X and Z. If $B_Y = 1 - \epsilon$ then for any $\epsilon \in (0, 1)$ the vector of endowments $E_X = 1$ and $E_i = 0$ for all $i \neq X$, and a vector of valuations $V = \{V_X = 1 - \epsilon/2, V_Z = 1, V_i = 0\}$ for all i besides X and Z results in inefficient allocation because the price at which Y resells the good to Z is $P(Y, Z) = V_Y + B_Y(V_Z - V_Y) = 1 - \epsilon$, such that the surplus of trade between X and Y is negative ($P(Y, Z) - V_X = -\epsilon/2$) and X keeps the good in equilibrium. If Y gets a lower resale price than the private valuation of X then the resale price of other agents who can intermediate between X and Z is even smaller because Y is the intermediary with the highest bargaining ability, and the private valuation of all potential intermediaries is zero. Therefore, if a trading network is incomplete then for any vector of bargaining ability B, exist vectors V and E such that the equilibrium is inefficient.

The intuition for this striking result is simple. When two agents cannot trade directly and all intermediaries receive less than full surplus when they resell an asset then it is not guaranteed that efficient trade will always happen because the private valuation of the seller can be above the resale price of the "strongest" intermediary but below the private valuation of the efficient buyer.

□

Proof of Proposition 4. If endowment of the good belongs to the agent with the highest valuation, he will keep it, and the equilibrium allocation is efficient. If endowment belongs to one of the agents with zero valuation, they sell it to the agent with the highest valuation, FB, or the second-highest valuation, SB. If SB sells in equilibrium then the equilibrium allocation will be efficient for all E. The price SB receives when he sells indirectly to FB is the highest when there is the smallest amount of intermediation, because prices decrease with the number of intermediaries. Let d represent the number of intermediaries on the shortest trading path between SB and FB. Let j be the first agent on the trading path between SB and FB. The equation for the price SB receives is:

$$P(SB, j) = \begin{cases} v + b(1 - v) & \text{if } j \text{ is } FB, \\ v + b(b^d - v) & \text{otherwise.} \end{cases} \tag{26}$$

Price is decreasing in d and therefore the highest price is achieved for the minimum number of intermediaries possible. If SB can sell directly to FB, the equilibrium is efficient because $P(SB, FB) = v + b(1 - v) > v$ for any $0 < b < 1$. If $j \neq FB$ then SB will sell if the surplus in trade is non-negative:

$$b^d > v. \tag{27}$$

The above condition states the resale value of the first intermediary is not lower than the private valuation of the agent with the second-highest valuation. This resale value is calculated by backward induction. The last intermediary sells the good for $0 + b(1 - 0) = b$. The next-to-the-last one sells it for $0 + b(b - 0) = b^2$, and so on. Therefore, if d intermediaries are required to transfer the good from SB to FB then the resale price of the first intermediary is b^d because in each trade, the price is multiplied by b. The solution to equation 27 provides the following condition on the maximum number of intermediaries needed for the equilibrium to be efficient for all E:

$$d \leq \hat{d} = \left\lfloor \frac{\log(v)}{\log(b)} \right\rfloor . \tag{28}$$

Given the number of intermediaries is an integer, $\hat{d}$ is rounded down to the closest integer.[53] If more than $\hat{d}$ intermediaries are required for SB to sell the good to FB then the equilibrium is inefficient. $\hat{d}$ is finite, because $0 < b < 1$ and $0 < v < 1$.

The worst case for the valuations vector in the simple economy is when the agent with the second-highest valuation needs the maximum number of intermediaries in the network to sell to the agent with the highest valuation. Therefore the equilibrium is efficient for all endowments and valuations in the simple economy if and only if any two agents require at most $\hat{d}$ intermediaries to trade in a trading network g. □

Proof of Proposition 5. Let $F(d)$ represent the probability that two randomly picked agents, i and j, in a homogeneous trading network need at least d intermediaries to trade. The probability that i and j can trade directly is p. Therefore the probability that i and j need at least one intermediary is

$$F(1) = (1 - p). \tag{29}$$

Agent i needs at least two intermediaries to trade with j if all i's trading partners need at least one intermediary to trade with j. The number of trading partners of i, k_i, is a random variable distributed according to binomial distribution. The (unconditional) probability that i and j need at least two intermediaries to trade is given by

$$F(2) = E[F(1)^{k_i}] = E[(1-p)^{k_i}] = (1-p^2)^{n-1} \geq (1-p)^{E[k_i]} = (1-p)^K. \tag{30}$$

The first inequality is implied by Jensen's inequality because $(1-p)^{k_i}$ is convex[54].

For $d = 3$, let k_i^2 be the number of agents who are trading partners of i's trading partners. Then $F(3)$ is given by

$$F(3) = E[(1-p)^{k_i^2}] \geq (1-p)^{E[k_i^2]} \geq (1-p)^{K^2}. \tag{31}$$

[53]Notation: if $\mathbb{Z}$ is the set of integers then $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$.

[54]The second derivative is $(1-p)^{k_i}(\log(1-p))^2 \geq 0$ for all $p \in [0, 1)$.

The first inequality is due to the convexity of $(1-p)^{k_i^2}$. The second inequality follows because of a possibility that some trading partners of i have the same trading partners. For example, if i has two trading partners and each of them has two trading partners then they may share one of the trading partners, such that $k_i^2 = 3$. The lower bound is calculated under the assumption that all trading partners of i's trading partners are different: $(1-p)^3 \geq (1-p)^4$. We can use the same approach for $d > 3$:

$$F(d) \geq (1-p)^{K^{d-1}} = (1 - \frac{K}{n-1})^{K^{d-1}}. \tag{32}$$

The intuition for the result is as follows. For i to need at least d intermediaries to trade with j, all of i's trading partners must need at least $d-1$ intermediaries to trade with j. Therefore $F(d) \geq F(d-1)^K$, whereas $F(1) = 1-p$.[55]

According to Proposition 4, the equilibrium allocation is efficient for all E if the agent with the second-highest valuation, SB, requires at most $\hat{d} = \left\lfloor \frac{\log(v)}{\log(b)} \right\rfloor$ intermediaries to trade with the agent with the highest valuation, FB. According to equation 32, the probability that SB requires more than $\hat{d}$ intermediaries to trade with FB is $F(\hat{d}+1) \geq (1-\frac{K}{n-1})^{K^{\hat{d}}}$. Therefore the probability that the equilibrium is efficient for all E in the simple economy is at most $1-(1-\frac{K}{n-1})^{K^{\hat{d}}}$ □

Proof of Theorem 1. According to Proposition 5, the upper bound on the probability that the equilibrium allocation is efficient for all E, is at most $1-(1-\frac{K}{n-1})^{K^{\hat{d}}}$, where $\hat{d} = \left\lfloor \frac{\log(v)}{\log(b)} \right\rfloor$. To prove the theorem, I take the limit of the upper bound for large n. If the upper bound of the probability tends to zero as the trading network grows then the probability tends to zero as well.

$$\lim_{n\to\infty} 1 - (1 - \frac{K}{n})^{K^{\hat{d}}}. \tag{33}$$

Case 1. The growth in the number of trading partners is of the same order as the

[55] A number of papers in physics study the average shortest path between nodes in different random graphs. Blondel, Guillaume, Hendrickx, and Jungers (2007) provide a recurrent formula for the probability that two randomly drawn nodes in the Erdős-Rényi random network are further than d links from each other, but this formula does not have a closed-form solution. They also point out that equation (12) in Fronczak, Fronczak, and Hołyst (2004), which was derived for more general random graphs, can be simplified in case of the Erdős-Rényi random network to $F(d) = e^{-1/n(np)^d}$. This solution, however, is not precise for high p. For example, it gives a 36 percent for the probability that two agents cannot trade directly in complete network, whereas the correct answer is 0 percent.

growth in the network size, such that $p \in (0, 1]$ is a constant:

$$\lim_{n\to\infty} 1 - (1-p)^{K^{\hat{d}}} = 1 - (1-p)^{\lim K^{\hat{d}}} = \begin{cases} p & \text{if } \hat{d} = 0 \\ 1 & \text{if } \hat{d} \geq 1. \end{cases} \tag{34}$$

Case 2. The growth in the number of trading partners is slower than the growth in the size of the network, $\lim_{n\to\infty} \frac{n}{K} = \infty$. In this case, first use the continuity of the exponential function:

$$\lim_{n\to\infty} (1 - \frac{K}{n})^{K^{\hat{d}}} = \lim_{n\to\infty} e^{K^{\hat{d}} \log(1-\frac{K}{n})} = e^{\left(\lim_{n\to\infty} K^{\hat{d}} \log(1-\frac{K}{n})\right)}. \tag{35}$$

Next rearrange the term inside the limit by dividing and multiplying by $-K/n$ and subtracting $\log(1) = 0$:

$$\lim_{n\to\infty} K^{\hat{d}} \log(1 - \frac{K}{n}) = \lim_{n\to\infty} \left(-\frac{K^{\hat{d}+1}}{n}\right) \left(\frac{\log(1-\frac{K}{n}) - \log(1)}{-\frac{K}{n}}\right). \tag{36}$$

Using the properties of limits,[56]

$$\lim_{n\to\infty} \left(-\frac{K^{\hat{d}+1}}{n}\right) \left(\frac{\log(1-\frac{K}{n}) - \log(1)}{-\frac{K}{n}}\right) = \lim_{n\to\infty} -\frac{K^{\hat{d}+1}}{n} \lim_{n\to\infty} \frac{\log(1-\frac{K}{n}) - \log(1)}{-\frac{K}{n}}. \tag{37}$$

The first limit depends on the rate of growth in the number of trading partners as network size increases:

$$\lim_{n\to\infty} -\frac{K^{\hat{d}+1}}{n} = -\left(\lim_{n\to\infty} \frac{K}{n^{1/(\hat{d}+1)}}\right)^{\hat{d}+1} = \begin{cases} 0 & \text{if } \lim_{n\to\infty} \frac{K}{n^{1/(\hat{d}+1)}} = 0 \\ -c^{\hat{d}} & \text{if } \lim_{n\to\infty} \frac{K}{n^{1/(\hat{d}+1)}} = c \\ -\infty & \text{if } \lim_{n\to\infty} \frac{K}{n^{1/(\hat{d}+1)}} = \infty. \end{cases} \tag{38}$$

The second limit is just a definition of the derivative of $\log(x)$ at $x = 1$. To see it, define $h = K/n$. Then we get

$$\lim_{n\to\infty} \frac{\log(1-\frac{K}{n}) - \log(1)}{-\frac{K}{n}} = \lim_{h\to 0} \frac{\log(1-h) - \log(1)}{-h} = \frac{d}{dx} \log(x)\,|_{x=1} = \frac{1}{x}\,|_{x=1} = 1. \tag{39}$$

[56]See Theorem 4.34 in Rudin (1976).

We can verify the product of the limits is well defined and therefore conditions for equation 37 are satisfied. Therefore the probability that the equilibrium is efficient for all E is at most

$$\lim_{n\to\infty} 1-(1-\frac{K}{n})^{K^{\hat{d}}} = \begin{cases} 1-e^{-0}=0 & \text{if } \lim_{n\to\infty} \frac{K}{n^{1/(\hat{d}+1)}} = 0. \\ 1-e^{-c^d} & \text{if } \lim_{n\to\infty} \frac{K}{n^{1/(\hat{d}+1)}} = c \\ 1-e^{-\infty}=1 & \text{if } \lim_{n\to\infty} \frac{K}{n^{1/(\hat{d}+1)}} = \infty \end{cases} \tag{40}$$

If the upper bound on the probability goes to zero in the limit then the probability goes to zero in the limit as well.

The last part of the proof requires verification that the network can still be connected even though the number of trading partners grows at a rate slower than $n^{1/(\hat{d}+1)}$. For a network to be connected with probability tending to 1 as $n \to \infty$, the probability of each trading relationship should be $p(n) > O(\log(n)/n)$ (Erdős and Rényi 1961; see Theorem 4.1 in Jackson (2008)). To show it for $K = n^{1/g}$, where $\hat{d}+1 < g < \infty$, the following condition should hold:

$$\lim_{n\to\infty} \frac{\log(n)}{n^{1/g}} = \lim_{n\to\infty} \frac{\log(n)^g}{n} = 0. \tag{41}$$

Applying L'Hôpital's rule g times and using the fact that g is finite we get

$$\lim_{n\to\infty} \frac{\log(n)^g}{n} = \lim_{n\to\infty} \frac{g\log(n)^{g-1}}{n} = \cdots = \lim_{n\to\infty} \frac{g!}{n} = 0. \tag{42}$$

Therefore, if the number of trading partners grows faster than $\log(n)$ but slower than $n^{1/(\hat{d}+1)}$ then the probability that the equilibrium allocation is efficient for all E tends to zero as the size of the trading network increases. To finish the proof, use the definition of $K = p(n-1)$ to get the condition on p, which is also the degree of network completeness. If $p(n) < O(\hat{p} \equiv n^{1/(\hat{d}+1)}/n = n^{-\hat{d}/(\hat{d}+1)}$ then the probability that the equilibrium allocation is efficient for all E tends to zero as the size of the trading network increases. □

Proof of Lemma 1. In the proof of Proposition 5, I have shown that the probability that two random agents require at least d intermediaries to trade in a homogeneous trading network is at least $(1-p)^{K^{d-1}}$. Therefore, the expected number of agents who are more than d trades from a randomly picked agent i is $(n-1)(1-p)^{K^{d-1}}$. The goal is to find d^{max} such that the expected number of agents further than d^{max} trades from i is less than one:

$$
\begin{aligned}
&(n-1)(1-p)^{K^{d-1}} < 1. \\
&(1-p)^{K^{d-1}} < \frac{1}{(n-1)} \\
&K^{d-1}\log(1-p) < -\log(n-1) \\
&(d-1)\log(K) > \log\left(\frac{-\log(n-1)}{\log(1-p)}\right) \\
&d > d^{max} \equiv \frac{\log\left(\frac{-\log(n-1)}{\log(1-p)}\right)}{\log(K)} + 1
\end{aligned}
$$

For $d > d^{max}$, the expected number of agents that are more than d trades from i is less than one, meaning an expected number of intermediaries is at least $D^h = \lfloor d_{max} \rfloor$. The expected number of intermediaries is non-negative because I assumed g is a connected network, such that $K > 1$.

This formula can be useful to calculate the diameter of a homogeneous trading network.[57] Diameter of any network is defined as the maximum shortest length path between any two nodes. If there are D intermediaries at most on the shortest path between any two agents then the diameter is $D + 1$. Therefore, the diameter of a homogeneous trading network is given by:[58]

$$
Diam^h(n,p) = \left\lfloor \frac{\log\left(\frac{-\log(n-1)}{\log(1-p)}\right)}{\log(K)} + 2 \right\rfloor . \tag{43}
$$

□

Proof of Proposition 6. The derivative of the expected welfare loss given in equation 20 with respect to b is

$$
\frac{\partial EWL^h(b,n,K,Q^h,G)}{\partial b} = \sum_{l=0}^{D-1} W(l,n,K)\left((1-b^l)G'(b^l)lb^{l-1} - (1-b^{l+1})G'(b^{l+1})(l+1)b^l\right), \tag{44}
$$

[57]Jackson (2008) writes that "Developing accurate estimates for diameters, even for completely random networks, turns out to be a formidable task that has been an active area of study in graph theory for the past four decades" (p. 105).

[58]Equation 43 is different from the usual approximation of this quantity used in the random networks literature, which is $Diam = \frac{\log(n)}{\log(K)}$ (see Fronczak, Fronczak, and Hołyst (2004)). For the case of $n = 1000$ and $p = n^{-0.5} \approx 3.16\%$, $Diam = 2$ while $Diam^h = 3$, which is consistent with unreported numerical results.

where $G'(v)$ is the density function of v and $W(l,n,K)$ is given by

$$W(l,n,K) \equiv F(l+1|n,K)\left(q_{SB} + q_{TB}\sum_{j=1}^{\infty} f(j|n,K)F(j+l)\right). \tag{45}$$

I rewrite the summation regrouping terms:

$$\frac{\partial EWL^h(b,n,K,Q^h,G)}{\partial b} = \tag{46}$$

$$\sum_{l=1}^{D-1} \underbrace{\left((1-b^l)G'(b^l)lb^{l-1}\right)}_{+} \underbrace{\left(-W(l-1,n,K)+W(l,n,K)\right)}_{-} - \underbrace{\left(W(D-1,n,K)(1-b^D)G'(b^D)Db^{D-1}\right)}_{+} < 0. \tag{47}$$

The terms in the first and last brackets are positive because $0 < b < 1$ and $G'(v)$ is non-negative because it is a density function. The second term is negative for all l because $W(l,n,K) \geq W(l+1,n,K)$ and $F(i|n,K) = F(i+1|n,K) + f(i+1|n,K) \geq F(i+1|n,K)$ for all n and K. Therefore the expected welfare loss is decreasing in b.[59]. □

7.5 Intermediation in a Homogeneous Trading Network

In this section, I compute $F(d|n,p)$—the probability that any two agents require at least d intermediaries to trade in a homogeneous trading network. As I stated in the proof of Proposition 5, the probability that two randomly picked agents require at least one intermediary to trade is $F(1|n,p) = 1-p$ because with probability p, they can trade directly. They need at least two intermediaries to trade if all trading partners of either of them require at least one intermediary to trade with the second agent. If agent i has k trading partners then the probability that he needs at least two intermediaries to trade with agent j equals $F(1)^k = (1-p)^k$, which is the probability that all trading partners of i can not trade directly with j. In a homogeneous trading network, the number of trading partners of each agent follows a binomial distribution because there are $n-1$ possible trading partners and p probability of a trading relationship with each one of them. Therefore, we can compute $F(2)$ by taking expectations of $(1-p)^{k_i}$, where k_i is the number of trading partners of agent i, which follows a binomial distribution with parameters $n-1$ and p. This computation gives us the formula for the probability that two randomly picked

[59] I assume the initial allocation is not always first best. If the initial allocation is first-best ($q_{TB} = q_{SB} = 0$) then the expected welfare loss is zero for any b.

agents require at least two intermediaries to trade:

$$F(2|n,p) = \sum_{j=0}^{n-1} (1-p)^j Pr(j|n-1,p) = (1-p^2)^{n-1}, \tag{48}$$

where $Pr(j|n-1,p) = \begin{pmatrix} n-1 \\ j \end{pmatrix} p^j (1-p)^{n-1-j}$ is the probability that an agent has j trading partners in a trading network with n agents. To compute $F(3|n,p)$, I use the same recursive approach as I used to compute $F(2|n,p)$. The idea is that if i and j require at least three intermediaries to trade, all trading partners of i require at least two intermediaries to trade with j.

$$F(3|n,p) = \sum_{j=0}^{n-1} F(2|n,p)^j Pr(j|n-1,p) = (1-p)^{n-1}(1+(1-p)^{n-2}p(1+p)^{n-1})^{n-1}. \tag{49}$$

In the proof of proposition 5, I highlight two reasons why $F(3) \geq (1-p)^{K^2}$. The first reason is the convexity of $(1-p)^{\alpha}$. The above formula solves this issue because we do not assume each agent has an average number of counterparties (sometimes referred as the mean-field approximation), but use the binomial distribution for the number of trading partners of each agent. The second reason for the inequality is that trading partners of an agent can share the same trading partners, and therefore my solution for $F(d>2|n,p)$ can still be biased downwards, meaning the true probability is larger. From the comparison of the analytical solution based on my formulas for $F(d)$ and the numerical solution presented in section 5.3, we can see my formulas for $F(d)$ allow me to compute the expected welfare loss in large network with high precision, suggesting the underestimation of $F(d>2)$ is not substantial for computing the expected welfare loss in large homogeneous trading networks.

Similarly, the probability that any two agents in a homogeneous trading network require at least four intermediaries to trade is computed as the probability that all trading partners of one of the agents require at least three intermediaries to trade with the second agent:

$$F(4|n,p) = \sum_{j=0}^{n-1} F(3|n,p)^j Pr(j|n-1,p) =$$

$$(1-p)^{n-1}\left(\frac{1+p\left(-1-p+p^2+(1-p^2)^n+(1-p)^n(1+p)\left(1+(1-p)^{n-2}p(1+p)^{n-1}\right)^n\right)}{(-1+p)^2(1+p)+p\left(1-p^2\right)^n}\right)^{n-1}. \tag{50}$$

Following a similar approach one can compute $F(d|n,p) = \sum_{j=0}^{n-1} F(d-1|n,p)^j Pr(j|n-1,p)$ for trading networks in which the maximum number of intermediaries is at least five.

This computation might be required for trading networks with tens of thousands of agents, all with a small number of counterparties. Alternatively, for $d \geq 5$, we can use the lower bound for $F(d)$ derived in the proof of proposition 5: $F(d) \geq (1-p)^{K^{d-1}}$.

Proof of Proposition 7. The proof is by contradiction. Assume there is a trading cycle in equilibrium, such that a group of $m \subseteq N$ agents trades between themselves in a cycle.[60] Each agent in this group sells the good an infinite number of times. I distinguish between an equilibrium in which the price in a trade between each pair of agents is the same and the case in which the price can be different. In the first equilibrium, we can focus on a single cycle in which some agent i sells the good to agent $i+1$, and agent $i+1$ sells to agent $i+2$ and so on till the good was sold again to agent i. First, trades happen only if the surplus is positive:

$$P(i+1, i+2) > V_i \text{ for all } i \in M. \tag{51}$$

Second, if each agent decides to sell rather than keep the good then following equation determines the price:

$$P(i, i+1) = \delta \max_2 P_{j \in N(i,g)} \text{ for all } i \in M. \tag{52}$$

From equations (51) and (52) and from the fact that $\delta \max_2 P_{j \in N(i,g)} < \max P_{j \in N(i,g)}$ for all i, it follows that

$$P(j+1, j+2) > P(j, j+1) \text{ for all } j \in M. \tag{53}$$

Given that inequality (53) holds for all trades in the cycle, it follows that $P(i, i+1) > P(i, i+1)$, which is a contradiction.

In the second case, prices are increasing with each transaction. However, with a finite budget constraint, agents will not be able to transact at infinite prices. Therefore, we can consider only a case in which an infinite strictly increasing sequence of prices $\{P_n\}_{n=1}^{\infty}$ converges to a finite price P ($\lim\limits_{n\to\infty} P_n = P$). The infinite sequence of prices converges to P; therefore, for any $\epsilon > 0$ there exists N such that $P - P_j < \epsilon$ for all $j > N$. Let P_n^i be the price for which agent i sells the good and $n > N$. Then $P - P_n^i < \epsilon$. Agent i sells to agent j who resells the good for P_{n+1}^j, such that $P - P_{n+1}^j < \epsilon$. An increase in price cannot

[60] Assume there are at least three agents in this trading cycle such that each agent has at least two trading partners. I prove that there are no trading cycles in the case of bilateral bargaining between two agents in proposition 1.

be larger than ϵ; otherwise, the higher price will be above P, so $P^j_{n+1} - P^i_n < \epsilon$. P^i_n is the discounted second-highest valuation among the trading partners of i, which is smaller than the discounted highest valuation. So we get, $P^j_{n+1} - \delta P^j_{n+1} \leq P^j_{n+1} - P^i_n < \epsilon$ or $P^j_{n+1} \leq \frac{1}{1-\delta}\epsilon$. Therefore, if $P < \epsilon + P^j_{n+1}$ then $P < \epsilon + \frac{1}{1-\delta}\epsilon$, which does not hold for every $\epsilon > 0$. For example, if $\epsilon = \frac{1-\delta}{2-\delta}P$ then we get $P < P$ which is a contradiction. Therefore, there are no trading cycles in equilibrium.

If there are no trading cycles in equilibrium then there is a non-empty set of agents who keep the good in equilibrium. Each trading path should eventually reach one agent in this set. □

Proof of Proposition 8. The proof is based on the contraction mapping theorem (see Stokey, Lucas, and Prescott (1989), Theorem 3.2). It is sufficient to show that M^a is a contraction mapping in a complete metric space (S, d), where $S = [0,1]^n$ and $d(x, y) = \max_{i \in N} |x_i - y_i|$ is a norm. This metric space is complete because it is a closed subset of the Euclidean space. Let $P, P' \in S$ be any two vectors of valuations. If there exists $\rho \in (0, 1)$ such that

$$\max_{i \in N}\{|M^a_i(P) - M^a_i(P')|\} \leq \rho \max_{i \in N}\{|P_i - P'_i|\} \tag{54}$$

then M^a is a contraction mapping. Next, I will prove that for $\rho = \delta$, the inequality (54) holds.[61]

Without loss of generality, let $l = \arg\max_{i \in N}\{|M^a_i(P) - M^a_i(P')|\}$.

Case 1: If $|M^a_l(P) - M^a_l(P')| = 0$ then the inequality holds.

Case 2: If $|M^a_l(P) - M^a_l(P')| = \delta|\max_2 P_{j \in N(l,g)} - \max_2 P'_{j \in N(l,g)}|$ then we need to consider a number of cases[62]:

Case 2.1: If $|M^a_l(P) - M^a_l(P')| = \delta|P_k - P'_k|$ then the inequality holds because $\delta|P_k - P'_k| < \rho \max_{i \in N}\{|P_i - P'_i|\}$. In this case, the same agent (k) has the second-highest valuation among l's trading partners both in vector P and vector P'. The inequality 54 holds in this case.

[61] If agents have different discount rates then we can define $\rho = \max_{i \in N}\{\delta_i\}$, where δ_i is the discount rate of agent i. The rest of the proof will not require any change.

[62] $\max_2 P'_{j \in N(l,g)}$ refers to the second-highest valuation among all trading partners of l in a trading network g when the vector of valuations is P'.

Case 2.2: Without loss of generality, assume $\max_2 P_j > \max_2 P'_j$ and let $k = \arg\max_2 P_{j\in N(l,g)}$. If $P'_k < \max_2 P'_{j\in N(l,g)}$ then $\max_2 P_{j\in N(l,g)} - \max_2 P'_{j\in N(l,g)} < P_k - P'_k \leq \max_{i\in N}\{|P_i - P'_i|\}$. The next to the last inequality follows from the fact that $P_k - P'_k = P_k - \max_2 P'_{j\in N(l,g)} + \max_2 P'_{j\in N(l,g)} - P'_k$ and $P'_k < \max_2 P'_{j\in N(l,g)}$. The inequality 54 holds in this case.

If $P'_k > \max_2 P'_j$ then let $h = \arg\max P_j$. If $\max_2 P'_j = P'_h$ then $\max_2 P_j - \max_2 P'_j < \max P_j - \max_2 P'_j < \max_{i\in N}\{|P_i - P'_i|\}$. If $\max_2 P'_j > P'_h$ then $\max_2 P_j - \max_2 P'_j \leq \max P_j - \max_2 P'_j < \max P_j - P'_h < \max_{i\in N}\{|P_i - P'_i|\}$. The inequality 54 holds in this case.

Case 3: If $\max_2 P_j > V_l$ and $\max_2 P'_j < V_l$ then $\max_2 P_j - V_l < \max_2 P'_j - \max_2 P_j < \max_{i\in N}\{|P_i - P'_i|\}$, where the last inequality follows from the proof of Case 2. The inequality 54 holds in this case.

The above analysis of all possible cases shows that inequality 54 holds when $\rho = \delta \in (0,1)$. Therefore, M^a is a contraction mapping and the vector of equilibrium valuations is unique. If several trading partners of i have the highest valuation then i can choose one of them randomly, in which case the vector of equilibrium trading decisions is generically unique. □

Proof of Proposition 9. Let $P, P' \in [0,1]^n$ be any two vectors of valuations. If there exists $\rho \in (0,1)$ such that

$$\max_{i\in N}\{|M^b_i(P) - M^b_i(P')|\} \leq \rho \max_{i\in N}\{|P_i - P'_i|\}, \tag{55}$$

then M^b is a contraction mapping in a metric space (S,d), where $S = [0,1]^n$ and $d(x,y) = \max_{i\in N}|x_i - y_i|$ is a norm.[63] I will prove that for $\rho = \max_{i\in N}\{B_i\}$, the inequality (55) holds.

Without loss of generality, let $l = \arg\max_{i\in N}\{|M^b_i(P) - M^b_i(P')|\}$. The inequality (55) holds if $|M^b_l(P) - M^b_l(P')| = 0$ because $\rho\max_{i\in N}\{|P_i - P'_i|\} \geq 0$. Further, I consider a number of cases for $|M^b_l(P) - M^b_l(P')| > 0$:

Case 1: Assume $M^b_l(P) = V_l + B_l(P_j - V_l)$ and $M^b_l(P') = V_l + B_l(P'_k - V_l)$, which is the case in which l sells to j when the vector of valuation is P, and he sells to k when the vector of valuation is P'. In this case, $|M^b_l(P) - M^b_l(P')| = B_l|P_j - P'_k|$. If j and k are the

[63] This metric space is complete because it is a closed subset of the Euclidean space. Therefore, proving M^b is a contraction mapping will allow me to rely on the contraction mapping theorem in solving for equilibrium valuations.

same agents then inequality (55) holds because $B_l|P_j - P'_j| \leq \rho \max_{i\in N}\{|P_i - P'_i|\}$. Further, I assume j and k are different agents. If $P_j = P'_k$ then the inequality (55) holds because $0 \leq \rho \max_{i\in N}\{|P_i - P'_i|\}$. If $P_j > P'_k$ then $B_l|P_j - P'_k| \leq B_l|P_j - P'_j| \leq \rho \max_{i\in N}\{|P_i - P'_i|\}$. The first inequality follows from the fact that $P'_k > P'_j$ because l always sells to the agent with the highest valuation among his direct trading partners. If $P_j < P'_k$ then $B_l|P'_k - P_j| \leq B_l|P'_k - P_k| \leq \rho \max_{i\in N}\{|P_i - P'_i|\}$. The first inequality follows from the fact that $P_j > P_k$ because l always sells to the agent with the highest valuation among his direct trading partners.

Case 2: Assume $M^b_l(P) = V_l + B_l(P_j - V_l)$ and $M^b_l(P') = V_l$, which is the case in which l sells to j when the vector of valuation is P, and keeps the good when the vector of valuation is P'. In this case $|M^b_l(P) - M^b_l(P')| = B_l|P_j - V_l|$. If he decides to keep the good, the implication is that $V_l > P'_j$, otherwise l would sell to j when the vector of valuations is P'. Therefore $B_l|P_j - V_l| \leq B_l|P_j - P'_j| \leq \rho \max_{i\in N}\{|P_i - P'_i|\}$.

Case 3: Assume $M^b_l(P') = V_l + B_l(P_j - V_l)$ and $M^b_l(P) = V_l$, which is the case in which l sells to j when the vector of valuation is P', and keeps the good when the vector of valuation is P. In this case, $|M^b_l(P) - M^b_l(P')| = B_l|P'_j - V_l|$. If he decides to keep the good, the implication is that $V_l > P_j$; otherwise, l would sell to j when the vector of valuations is P. Therefore, $B_l|P'_j - V_l| \leq B_l|P_j - P'_j| \leq \rho \max_{i\in N}\{|P_i - P'_i|\}$.

The above analysis of all possible cases shows that inequality 55 holds when $\rho = \max_{i\in N}\{B_i\} \in (0, 1)$ for any $P, P' \in S$. Therefore, M^b is a contraction mapping. $\square$