Rational Speculation in the Presence of Positive Feedback Traders: Stabilizing? Destabilizing? Or Neutral?

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Abstract

In the presence of positive feedback trading, there may be market overreaction, which is not removed but even accentuated by rational speculation. We examine the question of how likely this possibility is in a multi-period model. We show that overreaction can be ruled out under fairly general assumptions and tends to be weak otherwise and that rational speculation most likely stabilizes prices.

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1 Introduction

The theory of behavioral finance addresses the impact of rational speculation on equilibrium asset prices in the presence of irrational market participants in two different ways. In one strand of the literature (Shleifer and Vishny, 1997, Arnold, 2009), rational speculation stabilizes prices but is of limited efficacy, in that it possibly fails to bring prices to fundamental value. A second strand of the literature stresses that rational speculation may itself be destabilizing. The prototype model of this second setup was developed by De Long et al. (1990a, henceforth: DSSW). In their model, positive feedback traders exert positive demand in response to past price increases. Rational speculators bid up the asset price above the expected value of the asset's payoff, because they anticipate that they will be able to unload their holdings to non-rational agents before the overvaluation becomes apparent. In the absence of rational speculators, the price is equal to the expected value of the asset's payoff. So rational speculation is destabilizing in this model. In view of the prevalence of mechanisms that cause positive feedback effects, such as extrapolative expectations or stop-loss orders (cf. DSSW, Section I, pp. 381 ff.), this is an important result.

The present paper addresses the question of how general the result that rational speculation causes market overreaction, thereby destabilizing prices, is. To do so, we re-examine the DSSW model. The main difference compared to DSSW's model with three trading perdios is that we allow for a longer time horizon, so that we can consider different time patterns with regard to how long before the realization of the asset's payoffs a signal arrives and how feedback traders react to price increases in the recent and more distant past. The results raise some skepticism as to whether rational speculation causes overreaction and destabilizes prices. We show that there cannot be any market overreaction if the time span between the arrival of news and the realization of payoffs is longer than the time span over which changes in prices induce feedback trading. This casts doubt on whether the interaction of rational traders and positive feedback traders helps explain bubbles which start due to good news about an event in the more distant future and inflate over an extended period of time. Moreover, we argue that the magnitude of overreaction, if it occurs, tends to be low. This is because it is the feedback to one single price increase that determines the degree of overreaction. If large positions held by feedback traders are the sum of small reactions to many price changes, the feedback traders' positions are small and overreaction is weak. Finally, we show that if price increases further in the past have a weaker impact on feedback traders' demand, then rational speculation is stabilizing, rather that destabilizing.

The paper is organized as follows. Section 2 describes the model. Section 3 derives some useful results characterizing the agents' investment decisions. Sections 4-6 focus on illustrative special cases of the model. The general case is treated in Section 7. Section 8 concludes.

2 Model

We consider the DSSW model with one essential generalization: we allow for additional time periods in which no payoffs are realized and no information is revealed. We restrict attention to the case of a noiseless signal.

There are $T+2$ trading dates ($t = 0, 1, \dots, T+1$, $T \geq 2$) and three types of agents: a measure one of positive feedback traders, a measure μ of informed rational speculators, and a measure $1-\mu$ of passive investors, where $0 \leq \mu \leq 1$. Superscripts f, r, and e are used in order to distinguish variables referring to positive feedback traders, informed speculators, and passive investors, respectively.

There are one consumption good and one safe and one risky asset. The safe asset is in perfectly elastic supply. Its rate of return is zero. The supply S (≥ 0) of the risky asset is exogenous. The asset pays $v + \Phi + \theta$ at $T+1$, where v (≥ 0) is safe and Φ and θ are random variables with mean zero and positive, finite variances σ_Φ^2 and σ_θ^2, respectively. Φ is symmetrically distributed, and θ is independent of Φ. The realization of Φ becomes known to informed rational speculators at date T^r and to passive traders at date T^e ($0 < T^r < T^e \leq T$). No-one receives a signal about θ before $T+1$. The asset pays no dividends before $T+1$.

The three types of agents consume only at the final date $T+1$. Let p_t denote the price of the asset and D_t^i ($i = f, r, e$) the amount of the asset held by a type-i trader at date t ($= 0, 1, \dots, T$). The feedback traders' demand for the asset is $D_t^f = \delta$ at dates $t = 0, 1$ and

$$D_t^f = \delta + \sum_{l=1}^{t-1} \beta_l \Delta p_{t-l}, \quad t = 2, 3, \dots, T, \tag{1}$$

where $\Delta p_t = p_t - p_{t-1}$, $0 \leq \delta \leq S$, and $\beta_l \geq 0$ ($l = 1, 2, \dots, T-1$). That is, demand rises in response to past price increases. We often consider special cases of the model in which only the first feedback parameter β_1 ($= \beta$) is positive. For the greater part of the analysis, we further assume that $\delta = S$. However, we allow for $\delta \neq S$ to begin with, for this is essential in order to distinguish fundamental and non-fundamental risk. The passive investors' demand is

$$D_t^e = \alpha(v + E_t\Phi - p_t), \quad t = 0, 1, \dots, T. \tag{2}$$

$E_t\Phi$ is the expectation of Φ based on the observation of the signal at T^e, i.e., $E_t\Phi = 0$ for $t = 0, 1, \dots, T^e - 1$, and $E_t\Phi = \Phi$ for $t = T^e, T^e + 1, \dots, T$. That is, passive investors buy if they perceive that the dividend exceeds the asset price, and vice versa. They do not use the current price level to update their expectation of Φ. Following DSSW (p. 386), we assume that $\alpha = 1/(2\gamma\sigma_\theta^2) > \beta$.

This assumption ensures that the demand functions of rational speculators and passive investors are identical, once both have observed the signal. Informed rational speculators are the only utility maximizing agents in the model. Their preferences are represented by the mean-variance utility function $\mu - \gamma\sigma^2$ ($\gamma > 0$)), where μ and σ^2 are the mean and the variance of their final wealth, respectively.

Definition: *Prices p_t ($t = 0, 1, \ldots, T$) and demands D_t^f ($t = 2, 3, \ldots, T$) and D_t^i ($i = r, e$, $t = 0, 1, \ldots, T$) are an* equilibrium *if*

$\hookrightarrow$ *D_t^f satisfies (1) for all $t = 2, 3, \ldots, T$ and D_t^e satisfies (2) for all $t = 0, 1, \ldots, T$,*

$\hookrightarrow$ *D_t^r ($t = 0, 1, \ldots, T$) is the time-consistent solution to the mean-variance utility maximization problem, given current information, and*

$\hookrightarrow$ *the market for the risky asset clears:*

$$D_t^f + \mu D_t^r + (1 - \mu) D_t^e = S, \quad t = 0, 1, \ldots, T. \tag{3}$$

It is understood that prices and demands are random variables (i.e., functions of Φ) starting at date T^r (or T^e, if $\mu = 0$). The model analyzed by DSSW is the special case with $T = 2$, $T^r = 1$, $T^e = 2$, $v = 0$, $S = \delta = 0$, $\Phi \in \{-\phi, 0, \phi\}$ ($\phi > 0$), and θ distributed normally. In our preferred interpretation of the model, the sum of the lower bounds of the supports of Φ and θ is no less than $-v$, so that the asset's payoff is non-negative with certainty, and similar assumptions ensure that equilibrium prices are non-negative and arbitrage-free. Our preferred interpretation also entails that all agents hold initial wealth that is large enough to ensure non-negative consumption with certainty. As our preferred interpretation rules out normality of θ, we must not interpret the mean-variance utility function as an equivalent representation of a von Neumann-Morgenstern utility function as in DSSW (p. 384).[1]

3 Investment behavior

Let W_t^r denote a rational speculator's wealth at the beginning of t ($= 0, 1, \ldots, T+1$). Initial wealth W_0^r (> 0) is exogenous. Final wealth is

$$W_{T+1}^r = W_t^r + \sum_{t'=t}^{T-1} (p_{t'+1} - p_{t'}) D_{t'}^r + (v + \Phi + \theta - p_T) D_T^r, \quad t = 0, 1, \ldots, T-1. \tag{4}$$

[1]LW demonstrate that the use of mean-variance utility gives rise to equilibrium prices which are not arbitrage-free in De Long et al.'s (1990b) related model if, e.g., there is a lower bound on prices. We show below that their argument does not invalidate the present analysis (see footnote 4).

The second term on the right-hand side is the sum of the capital gains at dates $t' + 1$ through T. The final term on the right-hand side is the return on the date-T asset holdings. The rational speculators face a problem of dynamic mean-variance portfolio selection. The solutions are found recursively. The properties of the solution needed in the subsequent analysis are summarized in Lemmas 1-3.[2]

The first result states the well-known solution to the static final date mean-variance utility maximization problem. As emphasized by DSSW (pp. 385-6), the presence of dividend risk at date $T+1$ implies that the date-T demand is bounded.

Lemma 1: *A rational speculator's demand at T is*

$$D_T^r = \alpha(v + \Phi - p_T).$$

The next result states that if rational speculators can forecast the next-period asset price accurately, then arbitrage drives the current price to its anticipated next-period value. Let E_t^r denote the expectations operator conditional on rational investors' date-t information and $\sigma^{r2}_{p_{t+1}|t}$ the conditional variance of the date-$t+1$ price level.

Lemma 2: *If $\sigma^{r2}_{p_{t+1}|t} = 0$ for some t $(= 0, 1, \ldots, T-1)$, then*

$$p_{t+1} = p_t$$

in equilibrium, and the demand D_t^r is indeterminate.

Proof: Suppose $\sigma^{r2}_{p_{t+1}|t} = 0$ and $p_{t+1} \neq p_t$ for some t. Then, from (4), a rational speculator can achieve any level of final wealth W^r_{T+1} with certainty by choosing $D_t^r = (W^r_{T+1} - W^r_t)/(p_{t+1} - p_t)$ and $D^r_{t'} = 0$ for $t' = t+1, t+2, \ldots, T$. A solution to his date-t utility maximization problem does not exist. So $p_{t+1} = p_t$ must hold in equilibrium. From (4), W_T^r is then independent of D_t^r, so D_t^r is indeterminate. ||

The final result will be useful in determining the asset price in the date before rational investors observe the signal.

Lemma 3: *Suppose rational speculators have no posterior information about Φ at some t $(= 0, 1, \ldots, T-1)$, so that their subjective probability distribution for Φ is the unconditional distribution. Then in an equilibrium with $p_{t'} = v + (1+\lambda)\Phi$ $(t' = t+1, t+2, \ldots, T)$ for some λ $(\neq -1)$, they*

[2] There is no tractable "cookbook approach" to multi-period mean-variance optimization (cf. Li and Ng, 2000, and Steinbach, 2001, Section 2).

choose

$$D_t^r = \frac{E_t^r p_{t+1} - p_t}{2\gamma\sigma^{r2}_{p_{t+1}|t}} = \frac{v - p_t}{2\gamma(1+\lambda)^2\sigma_\Phi^2}.$$

Proof: From Lemmas 1 and 2 and the supposition that $p_{t'} = v + (1+\lambda)\Phi$ for $t' = t+1, t+2, \ldots, T$, (4) becomes

$$W_{T+1}^r = W_t^r + (p_{t+1} - p_t)D_t^r - \alpha\lambda\Phi(\theta - \lambda\Phi).$$

Since θ is independent of Φ,

$$E_t^r W_{T+1}^r = W_t^r + (E_t^r p_{t+1} - p_t)\, D_t^r + \alpha\lambda^2\sigma_\Phi^2.$$

Using $p_{t+1} = v + (1+\lambda)\Phi$, independence of θ, and the assumption that Φ is distributed symmetrically around zero (so that $E\Phi^3 = 0$), the conditional variance of final wealth $\sigma^{r2}_{W^r_{T+1}|t} = E_t^r(W^r_{T+1} - E_t^r W^r_{T+1})^2$ can be written as (see Appendix A.1)

$$\sigma^{r2}_{W^r_{T+1}|t} = \sigma^{r2}_{p_{t+1}|t}\,(D_t^r)^2 + (\alpha\lambda)^2\left(\sigma_\theta^2\sigma_\Phi^2 + \lambda^2\sigma^2_{\Phi^2}\right).$$

So maximization of mean-variance utility $E_t^r W^r_{T+1} - \gamma\sigma^{r2}_{W^r_{T+1}|t}$ is equivalent to maximizing $(E_t^r p_{t+1} - p_t)\, D_t^r - \gamma\sigma^{r2}_{p_{t+1}|t}(D_t^r)^2$. The expression for D_t^r in the lemma is the first-order condition for this problem. ||

4 Stabilizing rational speculation

Following DSSW (p. 388), we analyze the impact of rational speculation on equilibrium prices by comparing prices with and without rational informed traders (i.e., with $\mu > 0$ and $\mu = 0$). We first consider three special cases of the model, in which different price patterns with or without overreaction and with stabilizing, destabilizing, or neutral rational speculation occur. We then turn to the general case to investigate which properties of the examples are responsible for the different price patterns.

We start with the special case of the model with $T^r = 1$, $T^e = 2$, and $T = 3$. This is an example of stabilizing speculation.

Theorem 1: *Let $T^r = 1$, $T^e = 2$, $T = 3$, and $\beta_2 = 0$. For $\mu > 0$, the unique equilibrium asset price levels are $p_0 = v$ and*

$$p_t = v + \Phi = p^*(\Phi), \quad t = 1, 2, 3. \tag{5}$$

Proof: The model is solved backwards. At date 3, from (2) and Lemma 1, $D_3^i = \alpha(v + \Phi - p_3)$ for $i = r, e$. Together with (1), $\beta_1 = \beta$, $\beta_2 = 0$, and (3), we obtain

$$\alpha(v + \Phi - p_3) + \delta + \beta(p_2 - p_1) = S. \tag{6}$$

Rational speculators learn Φ at date 1. So from Lemma 2, $p_1 = p_2 = p_3$. From (6), it follows that

$$p_t = v + \Phi - \frac{S-\delta}{\alpha}, \quad t = 1, 2, 3. \tag{7}$$

Setting $S = \delta$ yields (5). As of date 0, neither informed nor passive traders have posterior information about Φ. From (5), $p_t = v + (1+\lambda)\Phi$ $(t = 1, 2, 3)$ for $\lambda = 0$, so Lemma 3 applies, and $D_0^r = (E_0^r p_1 - p_0)/(2\gamma\sigma_{p_1|0}^{r2})$. From (5), $E_0^r p_1 = v$ and the conditional price variance is $\sigma_{p_1|0}^{r2} = \sigma_\Phi^2$. Hence, $D_0^r = \alpha'(v - p_0)$, where $\alpha' = 1/(2\gamma\sigma_\Phi^2)$. Substituting D_0^r, $D_0^f = 0$, and $D_0^e = \alpha(v - p_0)$ into (3) yields $p_0 = v$. ||

Theorem 1 says that the asset price equals the expected value of the asset's payoff conditional on the rational speculators' current information. The risk premium for holding the risky asset is zero because the risk averse rational traders do not bear any risk. This is true for $t = 0, 1, 3$ since $D_t^r = S - D_t^f = S - \delta = 0$ and for $t = 2$ since $p_3 = p_2$. Note, however, that this is due to the assumption $S = \delta$. For $S > \delta$, from (7), a discount $(S-\delta)/\alpha$ is required in order to make the rational speculators and passive investors willing to hold the amount $S - \delta$ of the asset not held by the positive feedback traders. Following LW (Section II), the discount $(S-\delta)/\alpha$ should be interpreted as fundamental risk, even though it depends on the parameter δ from the feedback traders' demand function, because the feedback traders' demand has an impact on how much consumption risk rational traders have to bear in equilibrium. Keeping this in mind, we set $S = \delta$ from now on and define:

Definition: *Suppose $p_{T^r-1} = v$ for $\mu > 0$. Prices* overreact *(there is* overreaction*) if $p_t > p^*(\Phi)$ for $\Phi > 0$ and $p_t < p^*(\Phi)$ for $\Phi < 0$ for some $t \geq T^r$. Prices* overreact *for $\mu = 0$ if $p_{T^e-1} = v$ and $[p_t - p^*(\Phi)]\Phi > 0$ for some $t \geq T^e$.*

Definition: *Suppose $p_{T^r-1} = v$ for $\mu > 0$ and $p_{T^e-1} = v$ for $\mu = 0$. Rational speculation is* stabilizing *(or* destabilizing *or* neutral*) if $\max_{t \geq T^r} p_t$ for $\mu > 0$ is less than (or greater than or equal to, respectively) $\max_{t \geq T^e} p_t$ for $\mu = 0$ when $\Phi > 0$ (and vice versa when $\Phi < 0$).*

We also notice that, despite the fact that mean-variance optimizing choices do not necessarily exploit arbitrage opportunities, the equilibrium prices are arbitrage-free. With one risky and a safe asset, arbitrage-freeness requires that the returns of the two assets cannot be ranked according to first-order stochastic dominance. This condition is satisfied, since at the only date with uncertainty (i.e., at $t = 0$), the price of the risky asset can rise or fall. The conditions for non-negativity of prices and final wealth are in Appendix A.2.
Next, consider the model without informed rational speculators (i.e., for $\mu = 0$).

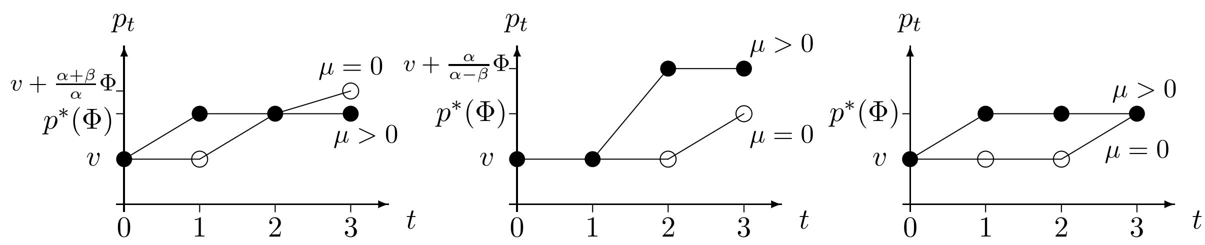

Figure 1: Price of the risky asset when $\Phi > 0$: stabilizing (left panel), destabilizing (middle panel), and neutral (right panel) rational speculation

Theorem 2: *Let $T^r = 1$, $T^e = 2$, $T = 3$, and $\beta_2 = 0$. For $\mu = 0$, the unique equilibrium asset prices are*

$$p_0 = p_1 = v, \quad p_2 = p^*(\Phi), \quad p_3 = v + \frac{\alpha + \beta}{\alpha}\Phi.$$

Proof: $p_0 = p_1 = v$ follows from $D_t^f = \delta$ $(t = 0, 1)$, (2), and (3). At date 2, passive investors learn Φ, and their demand is $D_2^e = \alpha(v + \Phi - p_2)$. The feedback traders' demand is $D_2^r = \delta$. Market clearing implies $p_2 = v + \Phi$. At date 3, the demands are $D_3^e = \alpha(v + \Phi - p_3)$ and $D_3^f = \delta + \beta\Phi$, respectively, and market clearing yields the final equality in the theorem. ||

The left panel of Figure 1 illustrates the price dynamics in response to a positive shock $\Phi > 0$ with and without informed rational speculators, i.e., with $\mu > 0$ and $\mu = 0$ respectively. Rational speculation ensures that the asset price is constant subsequent to the arrival of the signal and it precludes an increase above the asset's expected payoff when the information arrives. In the absence of rational speculators, the price increase at date 2, when the passive investors receive the positive signal, causes positive feedback trader demand at date 3, thereby driving the price above expected payoff. So there is overreaction in the absence of rational traders, and rational speculation is stabilizing.

5 Destabilizing rational speculation: the DSSW model

This section considers the DSSW model.

Theorem 3: *Let $T^r = 1$ and $T^e = T = 2$. For $\mu > 0$, the equilibrium prices are*

$$p_0 = v, \quad p_1 = p_2 = v + \frac{\alpha}{\alpha - \beta}\Phi. \tag{8}$$

For $\mu = 0$, the equilibrium prices are

$$p_0 = p_1 = v, \quad p_2 = p^*(\Phi). \tag{9}$$

Proof: First, let $\mu > 0$. From Lemma 1, $D_2^r = \alpha(v + \Phi - p_2)$. Using (1) and (2), the final-date market clearing condition becomes

$$\alpha(v + \Phi - p_2) + \beta(p_1 - p_0) = 0. \tag{10}$$

Since type-r investors learn Φ at date 1, Lemma 2 implies $p_2 = p_1$. From (10),

$$p_1 = \frac{\alpha(v + \Phi) - \beta p_0}{\alpha - \beta}. \tag{11}$$

Suppose $p_1 = v + (1 + \lambda)\Phi$ for some λ, so Lemma 3 yields $D_0^r = (E_0^r p_1 - p_0)/(2\gamma\sigma_{p_1|0}^{r2})$. Inserting the expressions for $E_0^r p_1$ and $\sigma_{p_1|0}^{r2}$ implied by (11) yields $D_0^r = (\alpha - \beta)\alpha'(v - p_0)/\alpha$. Together with $D_0^f = 0$ and $D_0^e = \alpha(v - p_0)$, it follows from (3) that $p_0 = v$. The validity of (8) follows from (11) and $p_1 = p_2$, thereby confirming that $p_1 = v + (1 + \lambda)\Phi$ for $\lambda = \beta/(\alpha - \beta)$.[3]
For $\mu = 0$, using (1) and (2), the market clearing condition (3) becomes $\alpha(v - p_t) = 0$ for $t = 0, 1$ and $\alpha(v + \Phi - p_2) = 0$, which immediately yields (9). ||

There is overreaction if, and only if, rational traders are present. So rational speculation is destabilizing here (see the middle panel of Figure 1). Rational traders buy assets from passive investors at date 1 and unload their holdings to feedback traders and take a short position at date 2. The date-2 price level has to exceed the asset's expected payoff by the amount necessary to compensate rational traders for the risk associated with the short position. This neatly formalizes the story how rational speculators benefit from market overreaction.[4]

[3]It can be shown that the equilibrium prices are unique. To do so, we have to allow for the possibility that $p_t \neq v$ for $\Phi = 0$. This makes the use of a generalized version of Lemma 3 necessary. For the sake of brevity, we skip the proof.

[4]LW's argument that equilibrium prices are not arbitrage-free in De Long et al.'s (1990b) model if, e.g., there is a lower bound on prices does not apply here. In De Long et al. (1990b), the interest rate on the safe asset is r (> 0) and the price p_t of the risky asset is i.i.d. It is easy to see that this generates an arbitrage opportunity if the support of p_t, $[\underline{p}, \bar{p}]$ say, is bounded. Investing p_t borrowed dollars in the risky asset yields the payoff $r + p_{t+1} - (1+r)p_t$. This is an arbitrage opportunity if $p_t \leq (r+\underline{p})/(1+r)$. Conversely, selling the risky asset short in t and investing the proceeds in the safe asset yields $(1+r)p_t - (r+p_{t+1})$ in $t+1$, and this is an arbitrage opportunity if $p_t \geq (r+\bar{p})/(1+r)$. The absence of arbitrage opportunities at all equilibrium price levels thus requires $\underline{p} > (r + \underline{p})/(1 + r)$ and $\bar{p} < (r + \bar{p})/(1 + r)$, i.e., $\underline{p} > 1 > \bar{p}$, a contradiction. The reason why LW's argument does not apply here is that the model considered is not a "DSSW-style model" in the sense of their Definition 2 (LW, p. 240), i.e., aggregate consumption and the asset price are not not perfectly correlated (cf. LW, Proposition 1, p. 241). To the contrary, aggregate consumption is zero at all dates at which the asset is traded and is, therefore, uncorrelated with the asset price.

6 Neutral rational speculation

The final special case we consider is $T^r = 1$ and $T^e = T = 3$. This model can be interpreted as the DSSW model with one additional date, in which "nothing happens" before the final date. This is an example of neutral rational speculation: the amplitude of the price change does not depend on when the news arrive (see the right panel of Figure 1).

Theorem 4: *Let $T^r = 1$, $T^e = T = 3$, and $\beta_2 = 0$. For $\mu > 0$, the unique equilibrium prices are*

$$p_0 = v, p_1 = p_2 = p_3 = p^*(\Phi). \tag{12}$$

For $\mu = 0$, the equilibrium prices are

$$p_0 = p_1 = p_2 = v, p_3 = p^*(\Phi). \tag{13}$$

Proof: The validity of (12) for $\mu > 0$ and of (13) for $\mu = 0$ follows from the same reasoning as in the proofs of Theorems 1 and 3, respectively. ||

7 The general case

We have seen that there may be overreaction in the presence of rational speculators or not and that rational speculation can be stabilizing, destabilizing, or neutral. What, then, are the characteristics of the special cases considered responsible for the occurrence of the respective cases? The distinguishing feature of the model with destabilizing rational speculation in Section 5 is that information arrives at the penultimate trading date, so that positive feedback trader demand is high at the final date, and rational speculators have to be compensated for the risk associated with the short position they have to take in order for the market to clear. The present section demonstrates that, more generally, overreaction occurs in the presence of rational speculators if *the feedback traders' demand at the final trading date reacts positively to the price change at the date when information about the risky asset's payoff arrives in the market.* Moreover, it is one single feedback parameter, viz. β_{T-T^r}, that determines the degree of overreaction.

Consider the general model outlined in Section 2 with $T \geq 2$ and $\beta_l \geq 0$.

Theorem 5: *For $\mu > 0$, the equilibrium prices are*

$$p_t = v, \quad t = 0, 1, \ldots, T^r - 1, \tag{14}$$

and

$$p_t = v + \frac{\alpha}{\alpha - \beta_{T-T^r}}\Phi, \quad t = T^r, T^r + 1, \ldots, T. \tag{15}$$

Proof: Market clearing at date T implies

$$\alpha(v+\Phi-p_T)+\sum_{l=1}^{T-1}\beta_l\Delta p_{T-l}=0. \tag{16}$$

Using Lemma 2 iteratively yields $p_{T^r}=p_{T^r+1}=\ldots=p_T$. Suppose $p_{T^r}=v+(1+\lambda)\Phi$. Then, from Lemma 3, the rational traders' asset demand at T^r-1 is $D^r_{T^r-1}=(v-p_{T^r-1})/[2\gamma(1+\lambda)^2\sigma^2_\Phi]$. Again applying Lemma 2 iteratively yields $p_0=p_1=\ldots=p_{T^r-1}$. So there is no positive feedback trading before T^r+1, and the date-T^r-1 market clearing condition becomes $\mu(v-p_{T^r-1})/[2\gamma(1+\lambda)^2\sigma^2_\Phi]+(1-\mu)\alpha(v-p_{T^r-1})=0$, so $p_{T^r-1}=v$. This proves (14). Substituting $p_{T^r}=p_T$, $p_{T^r-1}=v$, and $\Delta p_{T-l}=0$ for $l\neq T-T^r$ into (16) proves (15). ||

The fact that there is overreaction in the DSSW model (Section 5) but not in the models of Sections 4 and 6 is that $\beta_{T-T^r}=\beta>0$ in the former, whereas $\beta_{T-T^r}=\beta_2=0$ in the latter. In the absence of feedback trading (i.e., if $\beta_l=0$ for all $l=1,2,\ldots,t-1$ at all $t=2,3,\ldots,T$), (15) becomes $p_t=p^*(\Phi)$ for $t=T^r,T^r+1,\ldots,T$, and the proof is in essence Tirole's (1982, Proposition 3, pp. 1172-3) backward induction argument for a finite investment horizon.

Let $\beta_{\underline{L}}$ and $\beta_{\bar{L}}$ denote the first and the last strictly positive lag parameter in (1), respectively ($1\leq\underline{L}\leq\bar{L}\leq T-1$). There is no overreaction if either $T-T^r>\bar{L}$ or $T-T^r<\underline{L}$. The former condition, i.e., $T-T^r>\bar{L}$, is satisfied if the interval between the arrival of payoff-relevant news and the realization of the payoff exceeds the horizon of the feedback effects. This casts doubt on whether the model helps explain long-term price rallies, such as the conglomerate boom of the 1960s, the REIT boom of the 1970s (DSSW, p. 380), the dotcom boom of the 1990s, or the housing booms of the late 1990s and early 2000s. If rational traders anticipated the booms several years in advance, one has to argue that price increases several years ago affect positive feedback traders' demand. Otherwise the price would have been driven up to the expected payoff immediately after the positive news had arrived and remained constant then. The latter condition, i.e., $T-T^r<\underline{L}$, is satisfied when the rational speculators' informational advantage is sufficiently short-lived so that there is no time left for positive feedback effects before payoffs are realized. This casts doubt on whether the model helps explain excess volatility at very short horizons.

How strongly the asset price overreacts, if it overreacts, depends on the magnitude of β_{T-T^r}. There is little overreaction if β_{T-T^r} is small, so that $\alpha/(\alpha-\beta_{T-T^r})$ is close to one and the price level in (15) is close to $p^*(\Phi)$. It seems sensible to assume that the β_l's are small, if not zero, for l large. This reinforces the argument that the model's contribution to the explanation of long lasting booms is limited. Viewed from another angle, let $D^f_t=\delta$ for some t, and consider a sequence of asset price increases Δp (>0) over τ ($\geq\underline{L}$) dates. Let $\underline{L}=1$ and $\bar{L}\leq\tau$. From (1), the increase in the feedback traders' asset demand is

$$D^f_{t+\tau} - D^f_t = \Delta p \sum_{l=1}^{\bar{L}} \beta_l.$$

In order to explain a given increase in demand $D^f_{t'+\tau} - D^f_{t'}$, the individual feedback parameters β_l have to be smaller, the larger $\bar{L}$. If, for instance, the strictly positive feedback parameters are uniform, then they are inversely proportional to $\bar{L}$: $D^f_{t+\tau} - D^f_t = \Delta p \bar{L} \beta$ for $\beta_l = \beta$ $(l = 1, 2, \ldots, \bar{L})$. Generally, if price increases further back in the past have a weaker impact on demand (i.e., if β_l is a decreasing sequence), then the earlier information arrives, the lower the degree of overreaction.

Theorem 6: *Suppose $\beta_l = \beta$ for all $l = 1, 2, \ldots, T-1$. For $\mu = 0$, the unique equilibrium prices are*

$$p_t = v, \quad t = 0, 1, \ldots, T^e - 1, \tag{17}$$

and

$$p_t = v + \sum_{t'=T^e}^{t} \left(\frac{\beta}{\alpha}\right)^{t'-T^e} \Phi, \quad t = T^e, T^e + 1, \ldots, T. \tag{18}$$

Proof: For $t = 0, 1, \ldots, T^e - 1$, passive investors' demand is $D^e_t = \alpha(v - p_t)$, and feedback traders' demand is $D^f_t = \delta$. Market clearing implies (17). For $t = T^e, T^e + 1, \ldots, T$, the market clearing condition (6) can be expressed as

$$p_t = v + \Phi + \frac{1}{\alpha} \sum_{l=1}^{t-T^e} \beta \Delta p_{t-l}.$$

The validity of (18) follows upon substituting for p_t and the Δp_{t-l}'s into this equation. ||

Clearly, $p_t > p^*(\Phi)$ and $p_t < v + \sum_{t'=T^e}^{\infty} (\beta/\alpha)^{t'-T^e} \Phi = v + \alpha\Phi/(\alpha - \beta)$ for $\Phi > 0$ and $t > T^e$.

8 Conclusion

We have no doubt that the various kinds of feedback trading described by DSSW (Section I) are an important determinant of asset price volatility. The analysis in this paper raises doubts, however, about whether the model contributes significantly to its explanation.

Appendix A.1: Conditional variance of final wealth

In this Appendix, we derive the expression for the conditional variance of final wealth used in Section 3:

$$\sigma^{r2}_{W^r_{T+1}|t} = E^r_t \left(W^r_{T+1} - E^r_t W^r_{T+1}\right)^2$$

$$
\begin{aligned}
&= E_t^r\left\{(p_{t+1}-E_t^r p_{t+1})D_t^r-\alpha\lambda\left[\theta\Phi-\lambda\left(\Phi^2-E_t^r\Phi^2\right)\right]\right\}^2\\
&= E_t^r\Big\{(p_{t+1}-E_t^r p_{t+1})^2\,(D_t^r)^2\\
&\qquad+(\alpha\lambda)^2\left[\theta\Phi-\lambda\left(\Phi^2-E_t^r\Phi^2\right)\right]^2\\
&\qquad-2\alpha\lambda\underbrace{(p_{t+1}-E_t^r p_{t+1})}_{=(1+\lambda)\Phi}\left[\theta\Phi-\lambda\left(\Phi^2-E_t^r\Phi^2\right)\right]D_t^r\Big\}\\
&= \left[E_t^r\,(p_{t+1}-E_t^r p_{t+1})^2\right](D_t^r)^2\\
&\qquad+(\alpha\lambda)^2 E_t^r\left[(\theta\Phi)^2+\lambda^2\left(\Phi^2-E_t^r\Phi^2\right)^2-2\lambda\theta\Phi\left(\Phi^2-E_t^r\Phi^2\right)^2\right]\\
&\qquad-2\alpha\lambda(1+\lambda)\left[E_t^r\left(\theta\Phi^2\right)-\lambda E_t^r\left(\Phi^3-\Phi E_t^r\Phi^2\right)\right]D_t^r\\
&= \underbrace{\left[E_t^r\,(p_{t+1}-E_t^r p_{t+1})^2\right]}_{=\sigma^{r2}_{p_{t+1}|t}}(D_t^r)^2\\
&\quad+(\alpha\lambda)^2\Big\{\underbrace{E(\theta\Phi)^2}_{=\sigma_\theta^2\sigma_\Phi^2}+\lambda^2\underbrace{E\left(\Phi^2-E\Phi^2\right)^2}_{=\sigma^2_{\Phi^2}}-2\lambda\underbrace{E\left[\theta\Phi\left(\Phi^2-E\Phi^2\right)^2\right]}_{=0}\Big\}\\
&\qquad-2\alpha\lambda(1+\lambda)\Big\{\underbrace{E\left(\theta\Phi^2\right)}_{=0}-\lambda\Big[\underbrace{E\Phi^3}_{=0}-\underbrace{(E\Phi)}_{=0}\left(E\Phi^2\right)\Big]\Big\}D_t^r\\
&= \sigma^{r2}_{p_{t+1}|t}\,(D_t^r)^2+(\alpha\lambda)^2\left(\sigma_\theta^2\sigma_\Phi^2+\lambda^2\sigma^2_{\Phi^2}\right).
\end{aligned}
$$

Appendix A.2: Non-negativity conditions

For the sake of convenience, let $S=\delta$. Further, let $\underline{\theta}$ and $\bar{\theta}$ denote the lower and upper bounds of the support of θ, respectively, and analogously for Φ.

Non-negativity of $p^*(\Phi)$ for all Φ requires $\underline{\Phi}\geq -v$. This is implied by the condition that the asset has a non-negative payoff with certainty (i.e., $\underline{\Phi}+\underline{\theta}\geq -v$). The condition implies $\underline{\Phi}\geq -[(\alpha-\beta)/\alpha]v$ and, hence, $p^{**}(\Phi)\geq 0$ for all Φ.

Generally, since D_t^i does not affect wealth at dates with $p_t=p_{t+1}$ and D_t^i is bounded and independent of W_t^i at dates with price uncertainty, final wealth rises one-to-one with initial wealth and is non-negative with certainty if W_0^i is sufficiently large. For instance, in the model with $T^r=1$, $T^e=2$, and $T=3$, for $\mu=0$, the feedback traders hold the total supply of the risky asset both at the date before the price changes and at the final trading date (i.e., $D_1^f=D_3^f=\delta=S$), so the non-negativity condition for informed rational and passive traders is satisfied: $W_4^i=W_0^i>0$ $(i=r,e)$. Using $p_1-p_0=\Phi$ and $v+\Phi+\theta-p_3=\theta$, the positive feedback traders' final wealth is $W_4^f=W_0^f+(\Phi+\theta)\delta$. Non-negativity with certainty at the equilibrium prices requires

$W_0^f \geq -(\underline{\Phi} + \underline{\theta})\delta$. The final wealth levels with $\mu = 0$ are

$$W_4^f = W_0^f + \frac{\alpha}{\alpha - \beta}\delta\Phi + \left(\theta - \frac{\beta}{\alpha - \beta}\right)\Phi\left(\delta + \frac{\alpha\beta}{\alpha - \beta}\Phi\right)$$

and

$$W_4^i = W_0^i + \left(-\frac{\beta}{\alpha - \beta}\Phi + \theta\right)\left(-\frac{\alpha\beta}{\alpha - \beta}\Phi\right), \quad i = r, e.$$

As said above, final wealth is non-negative if initial wealth is sufficiently large.

Similar assumptions ensure non-negativity of prices and final wealth in the other variants of the model.

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